Preface

The Punjab Government established Punjab Technical University (PTU) in 1997 by an act of State Legislative. The University was entrusted with the responsibility of developing the new generation of technical manpower that can spearhead the industrial development of the State. Punjab Technical University has been envisaged to be the grooming ground for the future Engineers, Managers and Researchers.

As of today, PTU affiliates more than 300 Engineering, Management, Pharmacy, Hotel Management and Architecture institutions in the State that are approved by All India Council of Technical Education (AICTE).

PTU understands that restricting technical education to its campuses will not serve its objective of effective spreading of knowledge in the society. It is firmly understood that latest technical education has to be spread to the masses in every corner of the nation. This is how the Distance Education Programme (DEP) of the Punjab Technical University was conceived.

The objectives of the programme are to impart affordable, relevant, skill-based & remunerative technical education to the masses in the different corner of the country.

Today, the University has more than 2000 Learning Centres spread across the country offering quality technical education in the fields of Information Technology and Management, Paramedical Technology, Fashion Technology, Hotel Management and Tourism, Media and Mass Communication and Journalism etc.

The main purpose of this book is to impart the student an insight into the subject, explaining the complexities involved, in a simplified manner and helping them to achieve their academic goals.

For an easier navigation and understanding, this book contains the complete PTU curriculum of this subject and the topics. The various topics are dividing into Chapters, Units & Sub-Units and sufficient space is provided for students to make their brief notes.

This book encompasses a global approach for providing the simplified study material to both working as well as non-working students and is certain to get benefitted from the efforts of the authors of this book.

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This SIM has been prepared exclusively under the guidance of Punjab Technical University (PTU) and reviewed by experts and approved by the concerned statutory Board of Studies (BOS). It conforms to the syllabi and contents as approved by the BOS of PTU.
CAREER OPPORTUNITIES

The significance of the role played by Operations Research in the growth of any corporate or business entity cannot be overemphasized, as it forms a substantial part of the all-important quantitative methods employed by business concerns in decision making. Thus, students specializing in Operations Research stand to avail opportunities in the Production and Operations areas, in particular, and practically all the other departments including Finance, as a fair amount of knowledge of OR is required in all these areas.

A major source of employment is obviously the manufacturing industry where linear programming and quantitative techniques are employed while taking all decisions—big or small. Further, their capabilities and aptitude would be well—received where Logistics and Transportation are involved. In fact, the field of OR is indispensable for any concern. Thus, even in the hospitality industry there will be ample career opportunities for them. And, of course, academics and research are other areas where people with sound OR knowledge would be welcome.
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INTRODUCTION

Operational Research, or simply OR, originated in the context of military operations, but today it is widely accepted as a powerful tool for planning and decision-making, especially in business and industry. The OR approach has provided a new tool for managing conventional management problems. In fact, operational research techniques do constitute a scientific methodology of analysing the problems of the business world. They provide an improved basis for taking management decisions. The practice of OR helps in tackling intricate and complex problems such as that of resource allocation, product mix, inventory management, sequencing and scheduling, replacement, and a host of similar problems of modern business and industry.

With IT facilities becoming widely available, the significance and scope of OR has grown, and is still growing. Hence, OR is now an integral part of courses of computer science, economics, business management, public administration and several other disciplines.

Keeping all this in view, the book Operations Research introduces the basic elements of OR in the simplest possible way. The book has four units. Unit 1 describes the origin and development of operations research and its characteristics, besides elaborating on various models and methodology. Unit 2 explains Linear Programming, which is a decision-making technique. The unit describes various methods of solving a LP problem. Unit 3 deals with Transportation and Assignment Problems and the methods to find optimal solutions. Unit 4 explains the importance of decision-making and the situations in which decision-making plays a crucial role.

A Summary and Key Terms are provided at the end of each unit, for quick recollection. Besides, numerous examples are provided in the text. The ‘Check Your Progress’ questions and ‘Questions and Exercises’ sections will further help you in understanding the contents in a better way.
UNIT 1 INTRODUCTION TO OPERATIONS RESEARCH

Structure

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1.0 INTRODUCTION

Modern technological advance is accompanied by a growth of scientific techniques. While existing methods have been improved to meet the challenge of problems arising from the development of commerce and industry, a large number of new techniques or so-called sophisticated tools of analysis have been and are being devised to enlarge the extent of scientific knowledge to unlimited bounds of its applications. Such techniques have brought about a virtual revolution and can be reckoned as controlling forces in different walks of life. Operations Research, popularly known as OR, is a recent addition to a long list of scientific tools which provide a new outlook to many conventional management problems. OR adds greater sophistication in solving management problems. It seeks the determination of the best (optimum) course of action of a decision problem, given the limited resources. Therefore, OR has become a versatile tool in the field of management and its potential for future use is indeed substantial.

With the advent of automation, the centralization of organizational management has been disintegrated. In a single unit of industry, different departments of production, sales and inventory are governed by specialized agencies. Their targets and goals often disagree in an attempt to serve the common purpose of the organization. The policy decisions to coordinate these conflicting directives may be taken quite effectively, if an optimal solution can be determined from all available
alternative compromise formulae. OR provides an effective scientific technique to solve such decision-making problems of modern business and industry.

1.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Trace the origin and development of operations research
- Understand the various types of models in operations research and their advantages
- Describe the methodology of operations research
- Understand the role of decision-making in operations research
- Describe the application, use and limitations of operations research

1.2 ORIGIN AND DEVELOPMENT OF OPERATIONS RESEARCH

The subject of Operations Research (OR) was developed in military context during World War II, pioneered by the British scientists. At that time, the military management in England appointed a study group of scientists to deal with strategic and tactical problems related to air and land defence of the country. The main reason for conducting the study was that they were having very limited military resources. It was, therefore, necessary to decide upon the most effective way of utilizing these resources. As the name implies, operations research was apparently invented because the team was dealing with research on (military) operations. The scientists studied the various problems and on the basis of quantitative study of operations suggested certain approaches which showed remarkable success.

The encouraging results obtained by the British operations research teams consisting of personnel drawn from various fields like mathematics, physics, biology, psychology and other physical sciences, quickly motivated the United States military management to start similar activities. Successful innovations of the US teams included the development of new flight patterns, planning sea mining and effective utilization of electronic equipment. Similar OR teams also started functioning in Canada and France. These OR teams were usually assigned to the executive-in-charge of operations and as such their work came to be known as ‘Operational Research’ in the UK and by a variety of names in the United States: ‘operational analysis, operations evaluation, operations research, systems analysis, systems evaluation and systems research. The name ‘operational research’ or ‘operations research’ or simply OR is most widely used nowadays all over the world, for the new approach to systematic and scientific study of the operations of the system. Till the 1950s, use of operations research was mainly confined to military purposes.

After the end of World War II, the success of military teams attracted the attention of industrial managers who were seeking solutions to their complex managerial problems. At the end of the War, expenditures on defence research were reduced in the UK and this led to the release of many OR workers from the military at a time when industrial managers were confronted with the need to reconstruct most of
Britain’s manufacturing industries and plants that had been damaged in War. Executives in such industries sought assistance from the said OR workers. But in the USA, most war experienced operations research workers remained in military service as the defence research was increased and consequently, operations research was expanded at the end of the War. It was only in the early 1950s, that industry in the USA began to absorb the operations research workers under the pressure for increased demands for greater productivity originated because of the outbreak of the Korean conflict and because of technological developments in industry! Thus, operations research began to develop in industrial field in the United States since the year 1950. The Operations Research Society of America was formed in 1953 and the International Federation of Operations Research was formed in 1957.

Societies were established. Various journals relating to operations research began to appear in different countries in the years that followed the mid-1950s. Courses and curricula in operations research in different universities and other academic institutions began to proliferate in the United States. Other countries rapidly followed suit and thus, Operations Research came to be applied for solving business and industrial problems. Introduction of Electronic Data Processing (EDP) methods further enlarged the scope for application of OR techniques. With the help of a digited computer many complex problems can be studied on a day-to-day basis. As a result, many industrial concerns are adopting OR as an integrated decision-making tool for their routine decision procedures.

1.2.1 Nature and Characteristic Features of Operations Research (OR)

There have been various definitions for OR like applied decision-making, quantitative commonsense and making of economic decisions. The following definitions are generally talked about and widely accepted:

1. ‘OR is the application of scientific methods, techniques and tools to problems involving operations of systems so as to provide those in control of operations with optimum solutions to the problems’.
   — Churchman, Ackoff and Arnoff

2. ‘Operations Research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control’.
   — P.M. Morse and G.F. Kimball

3. ‘Operations Research is the attack of modern science on problems of likelihood that arise in the management and control of men and machines, materials and money in their natural environment, its special technique is to invent a strategy of control by measuring, comparing and predicting probable behaviour through a scientific model of a situation’.
   — Stafford Beer

4. ‘Operations Research is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive when he tries to achieve a thorough going rationality in dealing with his decision problems’.
   — D.W. Miller and M.K. Starr
From these definitions, we can state that Operational research can be considered as being the application of scientific method by interdisciplinary teams to problems involving the control of organized (man-machine) systems so as to provide solutions which best serve the purposes of the organization as a whole.

The different characteristics constituting the nature of OR can be summed up as follows:

1) **Interdisciplinary team approach**: Operations research has the characteristics that it is done by a team of scientists drawn from various disciplines such as mathematics, statistics, economics, engineering, and physics. It is essentially an interdisciplinary team approach. Each member of the OR team is benefited from the viewpoint of the other, so that a workable solution obtained through such a collaborative study has a greater chance of acceptance by management.

2) **Systems approach**: Operations research emphasizes on the overall approach to the system. This characteristic of OR is often referred to as system orientation. The orientation is based on the observation that in the organized systems the behaviour of any part ultimately has some effect on every other part. But all these effects are not significant and even not capable of detection. Therefore, the essence of system orientation lies in the systematic search for significant interactions in evaluating actions of any part of the organization. In O.R., an attempt is made to take account of all the significant effects and to evaluate them as a whole. OR thus considers the total system for getting the optimum decisions.

3) **Helpful in improving the quality of solution**: Operations research cannot give perfect answers or solutions to the problems. It merely gives bad answers to the problems which otherwise have worst answers. Thus, OR simply helps in improving the quality of the solution but does not result in perfect solution.

4) **Scientific method**: Operations research involves scientific and systematic attack of complex problems to arrive at the optimum solution. In other words, it uses techniques of scientific research. Thus it comprehends both aspects, i.e., it includes both scientific research on the phenomena of operating systems and the associated engineering activities aimed at applying the results of research.

5) **Goal-oriented optimum solution**: Operations research tries to optimize a well-defined function subject to given constraints and as such is concerned with the optimization theory.

6) **Use of models**: Operations research uses models built by quantitative measurement of the variables concerning a given problem and also derives a solution from the model using one or more of the diversified solution techniques. A solution may be extracted from a model either by conducting experiments on it or by mathematical analysis. The purpose is to help the management to determine its policy and actions scientifically.
(7) **Requires willing executives**: Operations research does require the willingness on the part of the executive for experimentation to evaluate the costs and the consequences of the alternative solutions of the problem. It enables the decision-maker to be objective in choosing an alternative from among many possible alternatives.

(8) **Reduces complexity**: Operations research tries to reduce the complexity of business operations and does help the executive in correcting a troublesome function and to consider innovations which are too costly and complicated to experiment with the actual practice. In view of this above, OR must be viewed both as a science and as an art. As science, OR provides mathematical techniques and algorithms for solving appropriate decision problems. OR is an art because success in all the phases that precede and succeed the solution of a problem largely depends on the creativity and personal ability of the decision-making analysts.

### 1.3 MODELS AND MODELLING IN OPERATIONS RESEARCH

A model in OR is a simplified representation of an operation, or is a process in which only the basic aspects or the most important features of a typical problem under investigation are considered. The objective of a model is to identify significant factors and interrelationships. The reliability of the solution obtained from a model depends on the validity of the model representing the real system.

A good model must possess the following characteristics:

(i) It should be capable of taking into account, new formulation without having any changes in its frame.

(ii) Assumptions made in the model should be as small as possible.

(iii) Variables used in the model must be less in number ensuring that it is simple and coherent.

(iv) It should be open to parametric type of treatment.

(v) It should not take much time in its construction for any problem.

#### 1.3.1 Advantages of a Model

There are certain significant advantages in using a model. These are:

(i) Problems under consideration become controllable.

(ii) It provides a logical and systematic approach to the problem.

(iii) It provides the limitations and scope of an activity.

(iv) It helps in finding useful tools that eliminate duplication of methods applied to solve problems.

(v) It helps in finding solutions for research and improvements in a system.

(vi) It provides an economic description and explanation of either the operation, or the systems it represents.
1.3.2 Classification of Models

The classification of models is a subjective problem. They may be distinguished as follows:

   (1) Models by degree of abstraction
   (2) Models by function
   (3) Models by structure
   (4) Models by nature of an environment
   (5) Models by the extent of generality

1.3.2.1 Models by function

These models consist of (a) Descriptive models (b) Predictive models and (c) Normative models.

   Descriptive models: They describe and predict facts and relationships among the various activities of the problem. These models do not have an objective function as part of the model to evaluate decision alternatives. In these models, it is possible to get information as to how one or more factors change as a result of changes in other factors.

   Normative or optimization models: They are prescriptive in nature and develop objective decision-rule for optimum solutions.

1.3.2.2 Models by structure

These models are represented by (a) Iconic models (b) Analogue models, and (c) Mathematical or symbolic models.

   Iconic or physical models: They are pictorial representations of real systems and have the appearance of the real thing. An iconic model is said to be scaled down or scaled up according to the dimensions of the model which may be smaller or greater than that of the real item, e.g., city maps, houses blueprints, globe, and so on. These models are easy to observe and describe, but are difficult to manipulate and are not very useful for the purpose of prediction.

   Analog models: These are more abstract than the iconic ones for there is no look alike correspondence between these models and real life items. The models in which one set of properties is used to represent another set of properties are called analog models. After the problem is solved, the solution is reinterpreted in terms of the original system. These models are less specific, less concrete, but easier to manipulate than iconic models.

   Mathematical or symbolic models: They are most abstract in nature. They employ a set of mathematical symbols to represent the components of the real system. These variables are related together by means of mathematical equations to describe the behaviour of the system. The solution of the problem is then obtained by applying well developed mathematical techniques to the model.

   The symbolic model is usually the easiest to manipulate experimentally and it is the most general and abstract. Its function is more explanatory than descriptive.
1.3.2.3 Models by nature of an environment

These models can be further classified into (a) Deterministic models and (b) Probabilistic models.

**Deterministic models:** They are those in which all parameters and functional relationships are assumed to be known with certainty when the decision is to be made. Linear programming and break-even models are the examples of deterministic models.

**Probabilistic or stochastic models:** These models are those in which at least one parameter or decision variable is a random variable. These models reflect to some extent the complexity of the real world and the uncertainty surrounding it.

1.3.2.4 Models by the extent of generality

These models can be further categorized into (a) Specific models (b) General models

When a model presents a system at some specific time, it is known as a specific model. In these models, if the time factor is not considered, they are termed as static models. An inventory problem of determining economic order quantity for the next period assuming that the demand in planning period would remain same as that of today is an example of static model. Dynamic programming may be considered as an example of dynamic model.

Simulation and Heuristic models fall under the category of general models. These models are used to explore alternative strategies which have been overlooked previously.

1.4 METHODOLOGY OF OPERATIONS RESEARCH

The methodology of OR generally involves the following steps:

1. **Formulating the Problem.** The first step in an OR study is to formulate the problem in an appropriate form. Formulating a problem consists in identifying, defining and specifying the measures of the components of a decision model. This means that all quantifiable factors which are pertinent to the functioning of the system under consideration are defined in mathematical language: variables (factors which are not controllable), and parameters or coefficient, along with the constraints on the variables and the determination of suitable measures of effectiveness.

2. **Constructing the Model.** The second step consists in constructing the model by which we mean that appropriate mathematical expressions are formulated which describe the interrelations of all variables and parameters. In addition, one or more equations or inequalities are required to express the fact that some or all of the controlled variables can only be manipulated within limits. Such equations or inequalities are termed as constraints or the restrictions. The model must also include an objective function which defines the measure of effectiveness of the system. The objective function and the constraints together constitute a model of the problem that we want to solve. This model describes the technology and the economics of the system under consideration through a set of simultaneous equations and inequalities.
(3) **Deriving the solution.** Once the model is constructed the next step in an OR study is that of obtaining the solution to the model, i.e., finding the optimal values of the controlled variables—values that produce the best performance of the system for specified values of the uncontrolled variables. In other words, an optimum solution is determined on the basis of the various equations of the model satisfying the given constraints and interrelations of the system and at the same time maximizing profit or minimizing cost or coming as close as possible to some other goal or criterion. How the solution can be derived depends on the nature of the model. In general, there are three methods available for the purpose viz., the analytical methods, the numerical methods and the simulation methods. Analytical methods involve expressions of the model by mathematical computations and the kind of mathematics required depends upon the nature of the model under consideration. This sort of mathematical analysis can be conducted only in some cases without any knowledge of the values of the variables, but in others the values of the variables must be known concretely or numerically. In latter cases, we use the numerical methods which are concerned with iterative procedures through the use of numerical computations at each step. The algorithm (or the set of computational rules) is started with a trial or initial solution and continued with a set of rules for improving it towards optimality. The initial solution is then replaced by the improved one and the process is repeated until no further improvement is possible. But in those cases where the analytical as well as the numerical methods cannot be used for deriving the solution, we use simulation methods, i.e., we conduct experiments on the model in which we select values of the uncontrolled variables with the relative frequencies dictated by their probability distributions. The simulation methods involve the use of probability and sampling concepts, and are generally used with the help of computers. Whichever method is used, our objective is to find an optimal or near-optimal solution, i.e., a solution which optimizes the measure of effectiveness in a model.

(4) **Testing the validity.** The solution values of the model, obtained as stated in step (3), are then tested against actual observations. In other words, effort is made to test the validity of the model used. A model is supposed to be valid if it can give a reliable prediction of the performance of the system represented through the model. If necessary, the model may be modified in the light of actual observations and the whole process is repeated till a satisfactory model is attained. The operational researcher quite often realizes that his model must be a good representation of the system and must correspond to reality which in turn requires this step of testing the validity of the model in an OR study. In effect, performance of the model must be compared with the policy or procedure that it is meant to replace.

(5) **Controlling the solution.** This step of an OR study establishes control over the solution by proper feedback of information on variables which might have deviated significantly. As such, the significant changes in
the system and its environment must be detected and the solution must accordingly be adjusted. This is particularly true when solutions are rules for repetitive decisions or decisions that extend over time.

(6) **Implementing the results.** Implementing the results constitutes the last step of an OR study. Because the objective of OR is not merely to produce reports but to improve the performance of systems, the results of the research must be implemented, if they are accepted by the decision makers. It is through this step that the ultimate test and evaluation of the research is made and it is in this phase of the study the researcher has the greatest opportunity for learning.

Thus, the procedure for an OR study generally involves some major steps viz., formulating the problem, constructing the mathematical model to represent the system under study, deriving a solution from the model, testing the model and the solution so derived, establishing controls over the solution and putting the solution to work-implementation. (Although the said phases and the steps are usually initiated in the order listed in an OR study, it should always be kept in mind that they are likely to overlap in time and to interact, i.e., each phase usually continues until the study is completed).

**Flow chart showing OR approach**

OR approach can also be illustrated through the following flow chart:
1.5 GENERAL METHODS FOR SOLVING OR MODELS

1.5.1 Operations Research and Decision-Making

Mathematical models have been constructed for OR problems and methods for solving the models are available in many cases. Such methods are usually termed as OR techniques. Some of the important OR techniques often used by decision-makers in modern times in business and industry are as follows:

(i) **Linear programming.** This technique is used in finding a solution for optimizing a given objective such as profit maximization or cost minimization under certain constraints. This technique is primarily concerned with the optimal allocation of limited resources for optimizing a given function. The name linear programming is because of the fact that the model in such cases consists of linear equations indicating linear relationship between the different variables of the system. Linear programming technique solves product-mix and distribution problems of business and industry. It is a technique used to allocate scarce resources in an optimum manner in problems of scheduling, product-mix, and so on. Key factors under this technique include an objective function, choice among several alternatives, limits or constraints stated in symbols and variables assumed to be linear.

(ii) **Waiting line or queuing theory.** Waiting line or queuing theory deals with mathematical study of queues. Queues are formed whenever the current demand for service exceeds the current capacity to provide that service. Waiting line technique concerns itself with the random arrival of customers at a service station where the facility is limited. Providing too much of capacity will mean idle time for servers and will lead to waste of money. On the other hand, if the queue becomes long, there will be a cost due to waiting of units in the queue. Waiting line theory, therefore, aims at minimizing the costs of both servicing and waiting. In other words, this technique is used to analyse the feasibility of adding facilities and to assess the amount and cost of waiting time. With its help we can find the optimal capacity to be installed which will lead to a sort of an economic balance between cost of service and cost of waiting.

(iii) **Inventory control/planning.** Inventory planning aims at optimizing inventory levels. Inventory may be defined as a useful idle resource which has economic value, e.g., raw-materials, spare parts, finished products, etc. Inventory planning, in fact, answers the two questions, viz., how much to buy and when to buy? Under this technique, the main emphasis is on minimizing costs associated with holding inventories, procurement of inventories and shortage of inventories.

(iv) **Game theory.** Game theory is used to determine the optimum strategy in a competitive situation. The simplest possible competitive situation is that of two persons playing zero-sum game, i.e., a situation in which two persons are involved and one person wins exactly what the other
loses. More complex competitive situations of the real life can also be imagined where game theory can be used to determine the optimum strategy.

(v) **Decision theory.** Decision theory concerns with making sound decisions under conditions of certainty, risk and uncertainty. As a matter of fact, there are three different types of states under which decisions are made, viz., deterministic, stochastic and uncertainty and the decision theory explains how to select a suitable strategy to achieve some object or goal under each of these three states.

(vi) **Network analysis.** Network analysis involves the determination of an optimum sequence of performing certain operations concerning some jobs in order to minimize overall time and/or cost. Programme Evaluation and Review Technique (PERT), Critical Path Method (CPM) and other network techniques such as Gantt Chart comes under Network Analysis. Key concepts under this technique are network of events and activities, resource allocation, time and cost considerations, network paths and critical paths.

(vii) **Simulation.** Simulation is a technique of testing a model which resembles a real life situation. This technique is used to imitate an operation prior to actual performance. Two methods of simulation are there: One is Monte Carlo method of simulation and the other is System Simulation Method. The former one using random numbers is used to solve problems which involve conditions of uncertainty and where mathematical formulation is impossible, but in case of System Simulation, there is a reproduction of the operating environment and the system allows for analysing the response from the environment to alternative management actions. This method draws samples from a real population instead of drawing samples from a table of random numbers.

(viii) **Integrated production models.** This technique aims at minimizing cost with respect to workforce, production and inventory. This technique is highly complex and is used only by big business and industrial units. This technique can be used only when sales and costs statistics for a considerable long period are available.

(ix) **Some other oR techniques.** In addition to these, there are several other techniques such as non-linear programming, dynamic programming, search theory, and the theory of replacement. A brief mention of some of these is as follows:

(a) **Non-linear programming** is that form of programming in which some or all the variables are curvilinear. In other words, this means that either the objective function or constraints or both are not in the linear form. In most practical situations, we encounter non-linear programming problems, but for computation purpose we approximate them as linear programming problems. Even then, there may remain some non-linear programming problems which may not be fully solved by presently known methods.

(b) **Dynamic programming** refers to the systematic search for optimal solutions to problems that involve many highly complex inter-
NOTES

relations that are, moreover, sensitive to multistage effects such as successive time phases.

(c) Heuristic programming, also known as discovery method, refers to step by step search toward an optimum when a problem cannot be expressed in the mathematical programming form. The search procedure examines successively a series of combinations that lead to stepwise improvements in the solution and the search stops when a near optimum has been found.

(d) Integer programming is a special form of linear programming in which the solution is required in terms of integral numbers (i.e., whole numbers) only.

(e) Algorithmic programming is just the opposite of heuristic programming. It may also be termed as near mathematical programming. This programming refers to a thorough and exhaustive mathematical approach to investigate all aspects of the given variables in order to obtain optimal solution.

(f) Quadratic programming refers to a modification of linear programming in which the objective equations appear in quadratic form, i.e., they contain squared terms.

(g) Parametric programming is the name given to linear programming when the later is modified for the purpose of inclusion of several objective equations with varying degrees of priority. The sensitivity of the solution to these variations is then studied.

(h) Probabilistic programming, also known as stochastic programming, refers to linear programming that includes an evaluation of relative risks and uncertainties in various alternatives of choice for management decisions.

(i) Search theory concerns itself with search problems. A search problem is characterized by the need for designing a procedure to collect information on the basis of which one or more decisions are made. This theory is useful in places in which some events are known to occur but the exact location is not known. The first search model was developed during the World War II to solve decision problems connected with air patrols and their search for submarines. Advertising agencies’ search for customers, personnel departments’ search for good executives are some examples of search theory’s application in business.

(j) The theory of replacement is concerned with the prediction of replacement costs and the determination of the most economic replacement policy. There are two types of replacement models—one type of models deal with replacing equipment that deteriorate with time and the other type of models helps in establishing replacement policy for those equipment which fail completely and instantaneously.
All these techniques are not simple but involve higher mathematics. The tendency today is to combine several of these techniques and form more sophisticated and advanced programming models.

1.6 APPLICATIONS AND LIMITATIONS OF OR

1.6.1 Applications of OR

Operations research (OR) has gained increasing importance since World War II in the technology of business and industry administration. It greatly helps in tackling the intricate and complex problems of modern business and industry. OR techniques are, in fact, examples of the use of scientific method of management. The following are some applications of OR:

(1) *OR provides a tool for scientific analysis.* OR provides the executives with a more precise description of the cause and effect relationship and the risks underlying the business operations in measurable terms and this eliminates the conventional intuitive and subjective basis on which managements used to formulate their decisions decades ago. In fact, OR replaces the intuitive and subjective approach of decision-making by an analytical and objective approach. The use of OR has transformed the conventional techniques of operational and investment problems in business and industry. As such, OR encourages and enforces disciplined thinking about organizational problems.

(2) *OR provides solution for various business problems.* OR techniques are being used in the field of production, procurement, marketing, finance and other allied fields. Problems like how best can managers and executives allocate the available resources to various products so that in a given time the profits are maximum or the cost is minimum? Is it possible for an industrial enterprise to arrange the time and quantity of orders of its stocks such that the overall profit with given resources is maximum? and How far is it within the competence of a business manager to determine the number of men and machines to be employed and used in such a manner that neither remains idle and at the same time the customer or the public has not to wait unduly long for service? can be solved with the help of OR techniques. Similarly, we might have a complex of industries—steel, machine tools and others—all employed in the production of one item, say, steel. At any particular time, we have a number of choices of allocating resources such as money, steel and tools for producing autos, building steel factories or tool factories. What should be the policy which optimizes the total number of autos produced over a given period? OR techniques are capable of providing an answer in such a situation.

Planning decisions in business and industry are largely governed by the picture of anticipated demands. The potential long-range profits of the business may vary in accordance with different possible demand patterns. The OR techniques serve to develop a scientific basis for coping with
the uncertainties of future demands. Thus, in dealing with the problem of uncertainty over future sales and demands, OR can be used to generate ‘a least risk’ plan.

At times there may be a problem of finding an acceptable definition of long-range company objectives. Management may be confronted with different viewpoints—some may stress the desirability of maximizing net profit, whereas others may focus attention primarily on the minimization of costs. OR techniques (especially that of mathematical programming such as linear programming) can help resolve such dilemmas by permitting systematic evaluation of the best strategies for attaining different objectives. These techniques can also be used for estimating the worth of technical innovations as also of potential profits associated with the possible changes in rules and policies.

How much changes can there be in the data on which a planning formulation is based without undermining the soundness of the plan itself? How accurately must managements know cost coefficients, production performance figures and other factors before it can make planning decisions with confidence? Much of the basic data required for the development of long-range plans is uncertain. Such uncertainties though cannot be avoided, but through various OR techniques, the management can know how critical such uncertainties are and this in itself is a great help to business planners.

(3) **OR enables proper deployment of resources.** OR renders valuable help in proper deployment of resources. For example, Programme Evaluation and Review Technique (PERT) enables us to determine the earliest and the latest times for each of the events and activities and thereby helps in the identification of the critical path. All this helps in the deployment of resources from one activity to another to enable the project completion on time. This technique, thus helps in determining the probability of completing an event or a project by a specified date.

(4) **OR helps in minimizing the waiting and servicing costs.** The waiting line or queuing theory helps the management in minimizing the total waiting and servicing costs. This technique also analyses the feasibility of adding facilities and thereby, helps the business people to take correct and profitable decision.

(5) **OR enables management to decide when to buy and how much to buy.** The main object of inventory planning is to achieve balance between the cost of holding stocks and the benefits from stock holding. Hence, the technique of inventory planning enables the management to decide when to buy and how much to buy.

(6) **OR assists in choosing an optimum strategy.** Game theory is specially used to determine the optimum strategy in a competitive situation and enables the businessmen to maximize profits or minimize losses by adopting the optimum strategy.

(7) **OR renders great help in optimum resource allocation.** Linear programming technique is used to allocate scarce resources in an optimum manner in problems of scheduling, product-mix, and so on.
This technique is popularly used by modern managements in resource allocation and in affecting optimal assignments.

(8) **OR facilitates the process of decision-making.** Decision theory enables the businessmen to select the best course of action when information is given in a probabilistic form. Through decision tree (a network showing the logical relationship between the different parts of a complex decision and the alternative courses of action in any phase of a decision situation) technique executive’s judgement can systematically be brought into the analysis of the problems. Simulation is another important technique used to imitate an operation or a process prior to actual performance. The significance of simulation lies in the fact that it enables in finding out the effect of alternative courses of action in a situation involving uncertainty where mathematical formulation is not possible. Even complex groups of variables can be handled through this technique.

(9) **Through OR, management can know the reactions of integrated business systems.** The Integrated Production Models technique is used to minimize cost with respect to workforce, production and inventory. This technique is quite complex and is usually used by companies having detailed information concerning their sales and costs statistics over a long period. Besides, various other OR techniques also help management people in taking decisions concerning various problems of business and industry. The techniques are designed to investigate how the integrated business system would react to variations in its component elements and/or external factors.

(10) **OR techniques help in the training/grooming of future (or would be) managers.** In fact, OR techniques substitute a means for improving the knowledge and skill of youngsters in the field of management.

### 1.6.2 OR and Modern Business Management

From what has been stated earlier, we can say that operational research renders valuable service in the field of business management. It ensures improvement in the quality of managerial decisions in all functional areas of management. The role of OR in business management can be summed up as follows:

1. **OR techniques help the directing authority** in optimum allocation of various limited resources viz., men, machines, money, material, time, etc., to different competing opportunities on an objective basis for achieving effectively the goal of a business unit. They help the chief of executive in broadening the management vision and perspectives in the choice of alternative strategies to the decision problems such as forecasting manpower, production capacities and capital requirements and plans for their acquisition.

2. **OR is useful to the production management** in (i) selecting the building site for a plant, scheduling and controlling its development and designing its layout; (ii) locating within the plant and controlling the movements of required production materials and finished goods inventories; (iii) scheduling and sequencing production by adequate preventive maintenance with optimum number of operatives by proper allocation of machines; and (iv) calculating the optimum product-mix.
3. **OR is useful to personnel management** to find out (i) the optimum manpower planning; (ii) the number of persons to be maintained on permanent or full time roll; (iii) the number of persons to be kept in a work pool intended for meeting the absenteeism; (iv) the optimum manner of sequencing and routing of personnel to a variety of jobs; and (v) in studying personnel recruiting procedures, accident rates and labour turnover.

4. **OR techniques equally help marketing management** to determine (i) where distribution points and warehousing should be located, their size, quantity to be stocked and the choice of customers; (ii) the optimum allocation of sales budget to direct selling and promotional expenses; (iii) the choice of different media of advertising and bidding strategies; and (iv) the consumer preferences relating to size, colour, and packaging, for various products as well as to outbid and outwit competitors.

5. **OR is also very useful to financial management** in (i) finding long-range capital requirements as well as how to generate these requirements; (ii) determining optimum replacement policies; (iii) working out a profit plan for the firm; (iv) developing capital-investments plans; and (v) estimating credit and investment risks.

In addition to all this, OR provides business executives with an understanding of business operations which gives them new insights and capability to determine better solutions for several decision-making problems with great speed, competence and confidence. When applied on the level of management where policies are formulated, OR assists the executives in an advisory capacity but on the operational level where production, personnel, purchasing, inventory and administrative decisions are made, it provides management with a means for handling and processing information. Thus, in brief; OR can be considered as a scientific method of providing executive departments with a quantitative basis for taking decisions regarding operations under their control.

### 1.6.3 Limitations of OR

OR—though it is a great aid to management as explained earlier—cannot be a substitute for decision-making. The choice of a criterion as to what is actually best for a business enterprise is still that of an executive who has to fall back upon his experience and judgement. This is so because of the several limitations of OR. Some important limitations are as follows:

1. **The inherent limitations concerning mathematical expressions.** OR involves the use of mathematical models, equations and similar other mathematical expressions. Assumptions are always incorporated in the derivation of an equation or model and such an equation or model may be correctly used for the solution of the business problems when the underlying assumptions and variables in the model are present in the concerning problem. If this caution is not given due care, then there always remains the possibility of wrong application of OR techniques. Quite often, operations researchers have been accused of having many solutions without being able to find problems that fit.

2. **High costs are incurred in the use of OR techniques.** OR techniques usually prove to be expensive. Services of specialized persons are
invariably called for (and along with it the use of computors) while using OR techniques. As such only big concerns can think of using such techniques. Even in big business organizations, we can expect that OR techniques will continue to be of limited use simply because they are not in many cases worth their cost. As opposed to this, a typical manager exercising intuition and judgement may be able to make a decision very inexpensively. Thus, the use of OR is a costlier affair and this constitutes an important limitation of OR.

(3) **OR does not take into consideration the intangible factors, i.e., non-measurable human factors.** OR makes no allowance for intangible factors such as skill, attitude, vigour of the management people in taking decisions but in many instances success or failure hinges upon the consideration of such non-measurable intangible factors. There cannot be any magic formula for getting an answer to management problems; much depends upon proper managerial attitudes and policies.

(4) **OR is only a tool of analysis and not the complete decision-making process.** It should always be kept in mind that OR alone cannot make the final decision. It is just a tool and simply suggests best alternatives, but in the final analysis many business decisions will involve human element. Thus, OR is at best a supplement rather than a substitute for management; subjective judgement is likely to remain a principal approach to decision-making.

(5) **Other limitations.** Among other limitations of OR, the following deserve mention:

(i) **Bias.** The operational researchers must be unbiased. An attempt to show results into a confirmation of management’s prior preferences can greatly increase the likelihood of failure.

(ii) **Inadequate objective functions.** The use of a single objective function is often an insufficient basis for decisions. Laws, regulations, public relations, market strategies, etc., may all serve to overrule a choice arrived at in this way.

(iii) **Internal resistance.** The implementation of an optimal decision may also confront internal obstacles such as trade unions or individual managers with strong preferences for other ways of doing the job.

(iv) **Competence.** Competent OR analysis calls for the careful specification of alternatives, a full comprehension of the underlying mathematical relationships and a huge mass of data. Formulation of an industrial problem to an OR set programme is quite often a difficult task.

(v) **Reliability of the prepared solution.** At times, a non-linear relationship is changed into linear for fitting the problem to the linear programming pattern. This may disturb the solution.

### 1.6.4 Frontiers/Scope of OR

OR techniques, having their origin with war operations analysis during World War II, have since assisted defence management to a larger extent. In defence
management, being vitally concerned with problems of logistics such as provisioning, distributing, search and forecasting, OR has given valuable help in decision formation. Although OR originated in context of military operations, its impact nowadays can be observed in many other areas. It has successfully entered into different research areas relating to modern business and industry. In fact, it is being reckoned as an effective scientific technique to solve several managerial decision-making problems such as inventory management problems, resource allocation problems, problems relating to replacement of both men as well as machines, sequencing and scheduling problems, queuing problems, competitive problems, search problems and the like ones.

The OR approach, though being practised by big organized business and industrial units, is equally applicable to small organizations. For example, whenever a departmental store face problems like employing additional sales girls and purchasing an additional van; we apply some techniques of operation research to minimize cost and maximize benefit for each such decision. As such it is often said that wherever there is a problem there is scope for applying operations research techniques.

In recent years of organized development, operations research has successfully entered, apart from its military and business applications, into several other areas of research. For instance, the basic problem in most of the developing countries in Asia and Africa is to remove poverty as quickly as possible. There is a considerable scope to solve this problem by applying operations research techniques. Similarly, with the explosion of population and consequent shortage of food, every country is facing the problem of optimum allocation of land for various crops in accordance with the climatic conditions and the available facilities. The problem of optimal distribution of water from a source like a canal for irrigation purposes is faced by each developing country. Hence, a good amount of scientific work related to the operations research can be done in this direction. Thus, there is considerable scope of applying operations research techniques for solving problems relating to national planning. National planning can be largely improved if an optimum coordination can be arrived through central planning agency by resorting to operations research models and techniques.

In the field of finance, the progress of operations research has been rather slow. Financial institutions have many situations similar to those prevailing in other commercial organizations. Investment policy must maximize the return on it, keeping the factor of risk below a specified level. Long-range corporate objectives of an institution are always studied along with functional objectives of individual departments, consistency between the two being essential for corporate planning. In particular, banking institutions require precise and accurate forecasting on cash management and capital budgeting. Linear programming models for many problems and quadratic programming formulation for some complex problems may prove the usefulness of operations research techniques to credit institutions.

The application of operations research techniques can as well be noticed in context of hospital management, health planning programmes, transportation system and in several other sectors. For instance, hospital management quite often faces the problem of allotting its limited resources of its multi-phased activities. Optimum allocation of resources so as to ensure a certain desirable level of service to patients can be reached through operations research. Similarly, for operating a transportation

Check Your Progress
4. What do you mean by formulating a problem?
5. What are the various steps involved in the procedure of an OR study?
6. Define integer programming.
7. Mention any two ways in which operations research helps personnel management.
service profitably, an optimum schedule for vehicles and crews becomes necessary. Punctuality, waiting time, total travel time, speed of the vehicles and agreement with trade unions of crews are some of the variables to be taken into account while preparing optimum schedules which can be worked out utilizing operations research techniques.

As stated earlier, operations research is generally concerned with problems that are tactical rather than strategic in nature, i.e., its use to long-range organizational planning problems is very much limited, but one can hope that with further developments taking place in the field, ‘Operations research should be able to deal with organizations in their entirety, rather than with slices through them. Managers who currently have to treat the whole with only intuition and experience as their guides, will then have the methods of science at their disposal as well’.1

This description brings home the point that during the last four decades the scope2 of operations research has extensively been widened and hopefully shall attain new strides in the days to come. The art of systems analysis, so well developed in the military context, will spread to other contexts, notably such civil government branches as criminal justice, urban problems, housing, health care, education and social services. There is growing awareness amongst the people that unless they make themselves familiar with operations research techniques, they would not be able to understand and appreciate the problems of modern business units. With computer facilities becoming widespread, the significance and scope of operations research is likely to grow in the coming years. In fact, the future holds great promise for the growth of OR in power, scope and practical importance and utility. OR will continue its currently vigorous efforts to reach out to new arenas of exploration and application.

1.7 SUMMARY

This unit introduced you to operations research, where you have learned about its origin and development of and its characteristics. The unit described the various models and modelling techniques applied in OR. Thus, you know that a model is a simplified representation of an operation or it is a process in which only the basic aspects or the most important features of a typical problem under investigation are considered. The various steps involved in the methodology of operations research involve formulating the problem, constructing the model, deriving the solution, testing the validity, controlling the solution and finally implementing the results. The unit also described various operations research techniques, most of which are used by organizations in decision-making. Thus, to mention a few advantages, operations research techniques help in the optimum allocation of various limited resources and are also useful to the production management in many ways. Also, you should have realized that though operations research is a great help to management, it cannot be a substitute for decision-making.

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1.8 KEY TERMS

- **Operations research**: It can be considered as being the application of scientific method by interdisciplinary teams to problems involving the control of organized (man-machine) systems so as to provide solutions which best serve the purposes of the organization as a whole.

- **Operations research model**: A model in OR is a simplified representation of an operation, or is a process in which only the basic aspects or the most important features of a typical problem under investigation are considered.

- **Descriptive models**: They describe and predict facts and relationships among the various activities of the problem. These models do not have an objective function as part of the model to evaluate decision alternatives.

- **Normative or optimization models**: They are prescriptive in nature and develop objective decision-rule for optimum solutions.

- **Iconic or physical models**: They are pictorial representations of real systems and have the appearance of the real thing. An iconic model is said to be scaled down or scaled up according to the dimensions of the model which may be smaller or greater than that of the real item, e.g., city maps, houses blueprints, globe, and so on.

- **Analog models**: Models in which one set of properties is used to represent another set of properties are called analog models.

- **Mathematic or symbolic models**: They are most abstract in nature. They employ a set of mathematical symbols to represent the components of the real system. These variables are related together by means of mathematical equations to describe the behaviour of the system.

- **Deterministic models**: They are those in which all parameters and functional relationships are assumed to be known with certainty when the decision is to be made. Linear programming and break-even models are the examples of deterministic models.

- **Probabilistic or stochastic models**: These models are those in which at least one parameter or decision variable is a random variable. These models reflect to some extent the complexity of the real world and the uncertainty surrounding it.

- **Queuing theory**: Waiting line or queuing theory deals with mathematical study of queues. The queues are formed whenever the current demand for service exceeds the current capacity to provide that service.

- **Simulation**: It is a technique of testing a model which resembles a real life situation. This technique is used to imitate an operation prior to actual performance.

- **Quadratic programming**: This refers to a modification of linear programming in which the objective equations appear in quadratic form, i.e., they contain squared terms.
1.9 ANSWERS TO ‘CHECK YOUR PROGRESS’

1. Operations research tries to optimize a well-defined function subject to given constraints and as such is concerned with the optimization theory.

2. (i) It provides a logical and systematic approach to the problem.
   (ii) Problems under consideration become controllable through a model.

3. Models are classified in the following ways: (a) Models by function, (b) Models by structure, (c) Models by degree of abstraction, (d) Models by nature of environment, (e) Models by the extent of generality.

4. Formulating a problem consists in identifying, defining and specifying the measures of the components of a decision model.

5. The various steps involve formulating the problem, constructing the mathematical model to represent the system under study, deriving a solution from the model, testing the model and the solution so derived, establishing controls over the solution and implementation.

6. Integer programming is a special form of linear programming in which the solution is required in terms of integral numbers only.

7. OR is useful to the personnel management in finding the optimum manpower requirement and in studying personnel recruiting procedures, accident rates and labour turnover.

1.10 QUESTIONS AND EXERCISES

Short-Answer Questions

1. Give two important definitions of OR.
2. Give a very brief account of the evolution of OR.
3. What are the advantages of a good OR model?
4. What are the different kinds of OR models based on structure?
5. Draw a flow chart showing the OR approach.

Long-Answer Questions

1. Define operations research and explain its main characteristics.
2. Describe in brief some of the important OR techniques used in modern business and industrial units.
3. Narrate fully the role of OR in the field of business and industry. Give examples in support of your answer.
4. Write a short note on ‘operations research’, describing some of its important techniques.
5. Discuss the limitations of OR techniques.
6. Do you think the day will come when all decisions in a business unit are made with the assistance of OR? Give reasons for your answer.
NOTES

7. ‘Operations research is a very powerful tool and analytical process that offers the presentation of an optimum solution in spite of its limitations.’ Discuss.

8. Describe the significance and scope of OR in modern business management.

9. Discuss the various phases in solving an OR problem.

10. Describe (i) the areas of application of OR, (ii) the state of OR in India.

1.11 FURTHER READING


UNIT 2  LINEAR PROGRAMMING

Structure

2.0  Introduction
2.1  Unit Objectives
2.2  Linear Programming Formulation
   2.2.1  General Formulation of LPP
   2.2.2  Matrix Form of LPP
2.3  Graphical Method
   2.3.1  Procedure for Solving LPP by Graphical Method
   2.3.2  General Formulation of LPP
   2.3.3  Matrix Form of LPP
   2.3.4  Some Important Definitions
   2.3.5  Canonical or Standard Forms of LPP
2.4  Big M Method
2.5  Simplex Method
   2.5.1  Introduction
   2.5.2  Definition
2.6  Duality in LP: Conversion of Primal to Dual
   2.6.1  Formulation of Dual Problem
   2.6.2  Definition of Dual Problem
   2.6.3  Important Results in Duality
2.7  Summary
2.8  Key Terms
2.9  Answers to ‘Check Your Progress’
2.10  Questions and Exercises
2.11  Further Reading

2.0  INTRODUCTION

Linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective functions. It is subject to a set of linear equalities and/or inequalities known as constraints. Linear programming is a mathematical technique which involves the allocation of limited resources in an optimal manner on the basis of a given criterion of optimality.

In this unit, the properties of Linear Programming Problems (LPP) are discussed. The graphical method of solving a LPP is applicable where two variables are involved. The most widely used method for solving LP problems consisting of any number of variables is called the simplex method developed by G. Dantzig in 1947 and made generally available in 1951.

2.1  UNIT OBJECTIVES

After going through this unit, you will be able to:

- Describe the procedure for mathematical formulation of LPP
- Explain the procedure for solving LPP by graphical method
- Describe the steps involved in solving LPP using the Big M method
2.2 LINEAR PROGRAMMING FORMULATION

The procedure for mathematical formulation of an LP problem consists of the following steps:

**Step 1** Write down the decision variables of the problem.

**Step 2** Formulate the objective function to be optimized (Maximized or Minimized) as a linear function of the decision variables.

**Step 3** Formulate the other conditions of the problem such as resource limitation, market constraints and interrelations between variables as linear inequations or equations in terms of the decision variables.

**Step 4** Add the non-negativity constraint from the considerations so that the negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraint and the non-negative constraint together form a linear programming problem.

### 2.2.1 General Formulation of LPP

The general formulation of the LPP can be stated as follows:

In order to find the values of \( n \) decision variables \( x_1, x_2, \ldots, x_n \) maximize or minimize the objective function.

\[
Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n
\]

\[
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1
\]

\[
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2
\]

\[
\vdots
\]

\[
a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq b_i
\]

\[
\vdots
\]

\[
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m
\]

where constraints may be in the form of inequality \( \leq \) or \( \geq \) or even in the form an equation (\( = \)) and finally satisfy the non negative restrictions

\[
x_1 \geq 0, x_2 \geq 0 \ldots x_n \geq 0
\]

### 2.2.2 Matrix Form of LPP

The LPP can be expressed in the matrix form as follows:

Maximize or

Minimize \( Z = Cx \rightarrow \) Objective function

subject to \( Ax \) (\( \leq \) or \( \geq \)) \( b \) Constant equation

\( b > 0, x \geq 0 \) Non-negativity restrictions,

where \( x = (x_1, x_2, \ldots, x_n) \)

\( c = (c_1, c_2, \ldots, c_n) \)
Example 2.1: A manufacturer produces two types of models $M_1$ and $M_2$. Each model of the type $M_1$ requires 4 hours of grinding and 2 hours of polishing; whereas each model of the type $M_2$ requires 2 hours of grinding and 5 hours of polishing. The manufacturers have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. Profit on $M_1$ model is Rs 3.00 and on model $M_2$ is Rs 4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a week?

Solution:

Decision variables: Let $X_1$ and $X_2$ be the number of units of $M_1$ and $M_2$ model.

Objective function: Since the profit on both the models are given, we have to maximize the profit, viz.

Max $Z = 3X_1 + 4X_2$

Constraints: There are two constraints, one for grinding and the other for polishing.

The number of hours available on each grinder for one week is 40 hrs. There are two grinders. Hence, the manufacturer does not have more than $2 \times 40 = 80$ hours of grinding. $M_1$ requires 4 hours of grinding and $M_2$ requires 2 hours of grinding.

The grinding constraint is given by

$4X_1 + 2X_2 \leq 80$

Since there are 3 polishers, the available time for polishing in a week is given by $3 \times 60 = 180$. $M_1$ requires 2 hours of polishing and $M_2$ requires 5 hours of polishing. Hence, we have $2X_1 + 5X_2 \leq 180$.

Finally we have

Max $Z = 3X_1 + 4X_2$

Subject to $4X_1 + 2X_2 \leq 80$

$2X_1 + 5X_2 \leq 180$

$x_1, x_2 \geq 0$

Example 2.2: A company manufactures two products $A$ and $B$. These products are processed in the same machine. It takes 10 minutes to process one unit of product $A$ and 2 minutes for each unit of product $B$ and the machine operates for a maximum of 35 hours in a week. Product $A$ requires 1 kg. and $B$ 0.5 kg. of raw material per unit the supply of which is 600 kg. per week. Market constraint on product $B$ is known to be 800 units every week. Product $A$ costs Rs 5 per unit and sold at Rs 10. Product $B$ costs Rs 6 per unit and can be sold in the market at a unit price of Rs 8. Determine the number of units of $A$ and $B$ per week to maximize the profit.


**Solution:**

**Decision variables:** Let \( x_1 \) and \( x_2 \) be the number of products \( A \) and \( B \).

**Objective function:** Costs of product \( A \) per unit is Rs 5 and sold at Rs 10 per unit.

\[
\therefore \text{ Profit on one unit of product } A = 10 - 5 = 5
\]

\[
\therefore x_1 \text{ units of product } A \text{ contributes a profit of Rs 5 profit contribution from one unit of product } B = 8 - 6 = 2
\]

\[
\therefore x_2 \text{ units of product } B \text{ contribute a profit of Rs 2 }
\]

\[
\therefore \text{ The objective function is given by}
\]

\[
\text{Max } Z \quad 5x_1 \quad 2x_2
\]

**Constraints:** Time requirement constraint is given by

\[
10x_1 \quad 2x_2 \quad (35 \quad 60)
\]

\[
10x_1 \quad 2x_2 \quad 21.00
\]

Raw material constraint is given by

\[
x_1 \quad 0.5x_2 \quad 600
\]

Market demand on product \( B \) is 800 units every week.

\[
\therefore x_2 \geq 800
\]

The complete LPP is

\[
\text{Max } Z \quad 5x_1 \quad 2x_2
\]

Subject to \( 10x_1 \quad 2x_2 \quad 2100 \)

\[
x_1 \quad 0.5x_2 \quad P \quad 600
\]

\[
x_2 \quad 800
\]

\[
x_1, x_2 \quad 0
\]

**Example 2.3:** A person requires 10, 12, and 12 units of chemicals \( A, B \) respectively for his garden. A liquid product contains 5, 2 and 1 units of \( A, B \) and \( C \) respectively per jar. A dry product contains 1, 2 and 4 units of \( A, B, C \) per carton. If the liquid product sells for Rs 3 per jar and the dry product sells for Rs 2 per carton, how many of each should be purchased, in order to minimize the cost and meet the requirements?

**Solution:**

**Decision variables:** Let \( X_1 \) and \( X_2 \) be the number of units of liquid and dry products.

**Objective function:** Since the cost for the products are given we have to minimize the cost.

\[
\text{Min } Z = 3x_1 + 2x_2
\]
**Constraints:** As there are three chemicals and its requirement are given, we have three constraints for these three chemicals.

\[
\begin{align*}
5x_1 & \quad x_2 & \quad 10 \\
2x_1 & \quad 2x_2 & \quad 12 \\
x_1 & \quad 4x_2 & \quad 12
\end{align*}
\]

Finally, the complete LPP is

\[\text{Min } Z = 3x_1 + 2x_2\]

**Example 2.4:** A paper mill produces two grades of paper, namely X and Y. Because of raw material restrictions, it cannot produce more than 400 tonnes of grade X and 300 tonnes of grade Y in a week. There are 160 production hours in a week. It requires 0.2 and 0.4 hours to produce a tonne of products X and Y respectively with corresponding profits of Rs 200 and Rs 500 per tonne. Formulate this as a LPP to maximize profit and find the optimum product mix.

**Solution:**

**Decision variables:** Let \(x_1\) and \(x_2\) be the number of units of two grades of paper X and Y.

**Objective function:** Since the profit for the two grades of paper X and Y are given, the objective function is to maximize the profit.

\[\text{Max } Z = 200x_1 + 500x_2\]

**Constraints:** There are 2 constraints one w.r.t. to raw material, and the other w.r.t. to concerning production hours.

\[\text{Max } Z = 200x_1 + 500x_2\]

**Example 2.5:** A company manufactures two products A and B. Each unit of B takes twice as long to produce as one unit of A and if the company were to produce only A it would have time to produce 2000 units per day. The availability of the raw material is sufficient to produce 1500 units per day of both A and B combined. Product B requiring a special ingredient only 600 units can be made per day. If A fetches a profit of Rs 2 per unit and B a profit of Rs 4 per unit, find the optimum product mix by graphical method.

**Solution:** Let \(x_1\) and \(x_2\) be the number of units of the products A and B respectively.

The profit after selling these two products is given by the objective function

\[\text{Max } Z = 2x_1 + 4x_2\]
Since the company can produce at the most 2000 units of the product in a day and type \( B \) requires twice as much time as that of type \( A \), production restriction is given by
\[ x_1 + 2x_2 \leq 2000 \]
Since the raw materials are sufficient to produce 1500 units per day both \( A \) and \( B \) combined we have \( x_1 + x_2 \leq 1500 \).
There are special ingredients for the product \( B \); we have \( x_2 \leq 600 \).
Also, since the company cannot produce negative quantities \( x_1 \geq 0 \) and \( x_2 \geq 0 \).
Hence, the problem can be finally put in the form:
Find \( x_1 \) and \( x_2 \) such that the profits\[ Z = 2x_1 + 4x_2 \]
\[ \text{is maximum.} \]
\[ x_1 + 2x_2 \leq 2000 \]
\[ x_1 + x_2 \leq 1500 \]
Subject to \( x_2 \leq 600 \)
\( x_1, x_2 \geq 0 \)

**Example 2.6:** A firm manufacturers 3 products \( A, B \) and \( C \). The profits are Rs 3, Rs 2 and Rs 4 respectively. The firm has 2 machines and the followign is the required processing time in minutes for each machine on each product.

<table>
<thead>
<tr>
<th>Product</th>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Machines</strong></td>
<td>( C )</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>( D )</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Machine \( C \) and \( D \) have 2000 and 2500 machine minutes respectively. The firm must manufacture 100 units of \( A \), 200 units of \( B \) and 50 units of \( C \) but no more than 150 \( A \)’s. Set up an LP problem to maximize the profit.

**Solution:** Let \( x_1, x_2, x_3 \) be the number of units of the product \( A, B, C \) respectively.
Since the profits are Rs 3, Rs 2 and Rs 4 respectively, the total profit gained by the firm after selling these three products is given by
\[ Z = 3x_1 + 2x_2 + 4x_3 \]
The total number of minutes required in producing these three products at machine \( C \) is given by \( 4x_1 + 3x_2 + 5x_3 \) and at machine \( D \) is given by \( 2x_1 + 2x_2 + 4x_3 \).
The restrictions on the machine \( C \) and \( D \) are given by 2000 minutes and 2500 minutes.
\[ 4x_1 + 3x_2 + 5x_3 \leq 2000 \]
\[ 2x_1 + 2x_2 + 4x_3 \leq 2500 \]
Also, since the firm manufactures 100 units of \( A \), 200 units of \( B \) and 50 units of \( C \) but not more than 150 unit of \( A \) the further restriction becomes
Hence, the allocation problem of the firm can be finally put in the form:

\[ \begin{align*}
\text{Find the value of } x_1, x_2, x_3 \text{ so as to maximize } \\
Z &= 3x_1 + 2x_2 + 4x_3 \\
\text{Subject to the constraints} \\
4x_1 &+ 3x_2 + 5x_3 \leq 2000 \\
2x_1 &+ 2x_2 + 4x_3 \leq 2500 \\
100 &\leq x_1, 150 \leq x_2, 0, 50 \leq x_3 \\
\end{align*} \]

**Example 2.7:** A farmer has a 100 acre farm. He can sell all tomatoes, lettuce or radishes and can raise the price to obtain Re 1.00 per kg. for tomatoes, Rs 0.75 a head for lettuce and Rs 2.00 per kg. for radishes. The average yield per acre is 2000 kg. of tomatoes, 3000 heads of lettuce and 1000 kgs of radishes. Fertilizers are available at Rs 0.50 per kg. and the amount required per acre is 100 kg each for tomatoes and lettuce and 50 kgs for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes and 6 man-days for lettuce. A total of 400 man-days of labour are available at Rs 20,000 per man-day. Formulate this problem as a linear programming model to maximize the farmer’s total profit.

**Solution:** Let \( x_1, x_2, x_3 \) be the area of his farm to grow tomatoes, lettuce and radishes respectively. The farmer produces \( 2000x_1 \) kgs of tomatoes, \( 3000x_2 \) heads of lettuce and \( 1000x_3 \) kg. of radishes.

\[ \begin{align*}
\text{\therefore The total sales of the farmer will be} & \quad = \text{Rs } (2000x_1 + 0.75 \times 3000x_2 + 2 \times 1000x_3) \\
\text{\therefore Fertilizer expenditure will be} & \quad = \text{Rs } 20 (5x_1 + 6x_2 + 5x_3) \\
\text{\therefore Farmer’s profit will be} & \quad Z = \text{Sale (in Rs)} - \text{total expenditure (in Rs)} \\
& = (2000x_1 + 0.75 \times 3000x_2 + 2 \times 1000x_3) - 0.5 \times [100(x_1 + x_2) + 50x_3] \\
& - 20 \times (5x_1 + 6x_2 + 5x_3) \\
& = 1850x_1 + 2080x_2 + 1875x_3 \\
\end{align*} \]

Since the total area of the farm is restricted to 100 acres
\[ x_1 + x_2 + x_3 \leq 100. \]

Also, the total man-days labour is restricted to 400 man-days
\[ 5x_1 \leq 6x_2 \leq 5x_3 \leq 400 \]

Hence, the farmer’s allocation problem can be finally put in the form:

\[ \begin{align*}
\text{Find the value of } x_1, x_2 \text{ and } x_3 \text{ so as to maximize} \\
Z &= 1850x_1 + 2080x_2 + 1875x_3
\end{align*} \]
Example 2.8: An electric appliance company produces two products: refrigerators and ranges. Production takes place in two separate departments I and II. Refrigerators are produced in department I and ranges in department II. The company’s two products are sold on a weekly basis. The weekly production can not exceed 25 refrigerators and 35 ranges. The company regularly employs a total of 60 workers in two departments. A refrigerator requires 2 man-weeks labour while a range requires 1 man-week labour. A refrigerator contributes a profit of Rs 60 and a range contributes a profit of Rs 40. How many units of refrigerators and ranges should the company produce to realize the maximum profit? Formulate this as a LPP.

Solution: Let \( x_1 \) and \( x_2 \) be the number of units of refrigerators and the ranges to be produced.

Each refrigerator and range contributes a profit of Rs 60 and Rs 40 respectively. The objective function is to maximize \( Z = 60x_1 + 40x_2 \).

There are 2 constraints which are imposed on weekly production and labours.

Since the weekly production cannot exceed 25 refrigerators and 35 ranges,

\[
\begin{align*}
1 & \leq 25 \\
1 & \leq 35 
\end{align*}
\]

A refrigerator requires 2 man-weeks of labour and a range requires 1 man-week of labour and the total number of workers is 60.

\[ 2x_1 + x_2 \leq 60. \]

Non-negativity restrictions. Since the number of refrigerators and ranges produced cannot be negative, we have \( x_1 \geq 0 \) and \( x_2 \geq 0 \).

Hence, the production of refrigerator and ranges problem can be finally put in the form:

Find the value of \( x_1 \) and \( x_2 \) so as to maximize

\[ Z = 60x_1 + 40x_2. \]

Subject to

\[
\begin{align*}
x_1 & \geq 25 \\
x_2 & \geq 35 \\
2x_1 + x_2 & \leq 60 \\
and & x_1, x_2 \geq 0
\end{align*}
\]

2.3 GRAPHICAL METHOD

A simple linear programming problem with two decision variables can be easily solved by the graphical method.
2.3.1 Procedure for Solving LPP by Graphical Method

The steps involved in the graphical method are as follows.

**Step 1** Consider each inequality constraint as an equation.

**Step 2** Plot each equation on the graph as each will geometrically represent a straight line.

**Step 3** Mark the region. If the inequality constraint corresponding to that line is \( O \) then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality constraint \( P \) sign, the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region, thus obtained, is called the feasible region.

**Step 4** Assign an arbitrary value, say zero, for the objective function.

**Step 5** Draw a straight line to represent the objective function with the arbitrary value (i.e., a straight line through the origin).

**Step 6** Stretch the objective function line till the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin, passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passes through at least one corner of the feasible region.

**Step 7** Find the coordinates of the extreme points selected in step 6 and find the maximum or minimum value of \( Z \).

**Note:** As the optimal values occur at the corner points of the feasible region, it is enough to calculate the value of the objective function of the corner points of the feasible region and select the one which gives the optimal solution, i.e., in the case of maximization problem the optimal point corresponds to the corner point at which the objective function has a maximum value and in the case of minimization, the corner point which gives the objective function the minimum value is the optimal solution.

**Example 2.9:** Solve the following LPP by graphical method.

Minimize \( Z = 20X_1 + 10X_2 \)

Subject to

\[ X_1 + 2X_2 \leq 40 \]
\[ 3X_1 + X_2 \geq 30 \]
\[ 4X_1 + 3X_2 \geq 60 \]
\[ X_1, X_2 \geq 0 \]

**Solution:** Replace all the inequalities of the constraints by equation

\[ X_1 + 2X_2 = 40 \quad \text{If } X_1 = 0 \Rightarrow X_2 = 20 \]
\[ \quad \text{If } X_2 = 0 \Rightarrow X_1 = 40 \]
\[ \therefore X_1 + 2X_2 = 40 \quad \text{passes through (0, 20) (40, 0)} \]
\[ 3X_1 + X_2 = 30 \quad \text{passes through (0, 30) (10, 0)} \]
\[ 4X_1 + 3X_2 = 60 \quad \text{passes through (0, 20) (15, 0)} \]

Plot each equation on the graph.
The feasible region is $ABCD$.

$C$ and $D$ are the points of intersection of lines

$X_1 + 2X_2 = 40$, $3X_1 + X_2 = 30$ and

$4X_1 + 3X_2 = 60$, $X_1 + X_2 = 30$

On solving we get $C = (4, 18)$ $D = (6, 12)$

Corner points Value of $Z = 20X_1 + 10X_2$

$A (15, 0)$ 300
$B (40, 0)$ 800
$C(4, 18)$ 260
$D (6, 12)$ 240 (Minimum value)

∴ The minimum value of $Z$ occurs at $D (6, 12)$. Hence, the optimal solution is $X_1 = 6, X_2 = 12$.

**Example 2.10:** Find the maximum value of $Z = 5X_1 + 7X_2$.

Subject to the constraints,

$X_1 + X_2 \leq 4$

$3X_1 + 8X_2 \leq 24$

$10X_1 + 7X_2 \leq 35$

$X_1, X_2 > 0$

**Solution:** Replace all the inequalities of the constraints by forming equations.

$X_1 + X_2 = 4$ passes through $(0, 4)$ $(4, 0)$

$3X_1 + 8X_2 = 24$ passes through $(0, 3)$ $(8, 0)$

$10X_1 + 7X_2 = 35$ passes through $(0, 5)$ $(3.5, 0)$

Plot these lines in the graph and mark the region below the line since the inequality of the constraint is $\leq$ and is also lying in the first quadrant.
The feasible region is $OABCD$. B and C are points of intersection of lines.

$X_1 + X_2 = 4, \ 10X_1 + 7X_2 = 35$ and

$3X_1 + 8X_2 = 24, \ X_1 + X_2 = 4$

On solving we get,

B = (1.6, 2.3)

C = (1.6, 2.4)

<table>
<thead>
<tr>
<th>Corner points</th>
<th>Value of $Z = 5X_1 + 7X_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>O (0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>A (3.5, 0)</td>
<td>17.5</td>
</tr>
<tr>
<td>B (1.6, 2.3)</td>
<td>25.1</td>
</tr>
<tr>
<td>C (1.6, 2.4)</td>
<td>24.8 (Maximum value)</td>
</tr>
<tr>
<td>D (0, 3)</td>
<td>21</td>
</tr>
</tbody>
</table>

∴ The maximum value of $Z$ occurs at C (1.6, 2.4) and the optimal solution is $X_1 = 1.6, X_2 = 2.4$.

**Example 2.11:** A company produces 2 types of hats. Every hat of type $A$ requires twice as much labour time than the second type $B$. If the company produces only hat $B$ then it can produce a total of 500 hats a day. The market limits daily sales of the hat $A$ and hat $B$ to 150 and 250 hats. The profits on hat $A$ and $B$ are Rs 8 and Rs 5 respectively. Solve graphically to get the optimal solution.

**Solution:** Let $X_1$ and $X_2$ be the number of units of type $A$ and type $B$ hats respectively.

Max $Z = 8X_1 + 5X_2$

Subject to $2X_1 + 2X_2 \leq 500$

$X_1 \geq 150$

$X_2 \geq 250$

$X_1, X_2 \geq 0$.

First rewrite the inequality of the constraint into an equation and plot the lines in the graph.
\[ 2X_1 + X_2 = 500 \quad \text{passes through (0, 500) (250, 0)} \]
\[ X_1 = 150 \quad \text{passes through (150, 0)} \]
\[ X_2 = 250 \quad \text{passes through (0, 250)} \]

**NOTES**

We mark the region below the lines lying in the first quadrant since the inequality of the constraints is \( \leq \). The feasible region is \( OABCD \) and \( B \) and \( C \) are points of intersection of lines

\[ 2X_1 + X_2 = 500, \quad X_1 = 150 \quad \text{and} \]
\[ 2X_1 + X_2 = 500, \quad X_2 = 250. \]

On solving, we get \( B = (150, 200) \)
\[ C = (125, 250) \]

![Graph showing feasible region and corner points]

**Corner points**

Value of \( Z = 8X_1 + 5X_2 \)

\begin{align*}
O & (0,0) & 0 \\
A & (150,0) & 1200 \\
B & (150,200) & 2200 \\
C & (125,250) & 2250 \quad \text{(Maximum Z = 2250)} \\
D & (0,250) & 1250
\end{align*}

The maximum value of \( Z \) is attained at \( C \) (125, 250).

\[ \therefore \quad \text{The optimal solution is } X_1 = 125, \ X_2 = 250. \]

i.e., the company should produce 125 hats of type A and 250 hats of type B in order to get the maximum profit of Rs 2250.

**Example 2.12:** By graphical method, solve the following LPP.

Max \( Z = 3X_1 + 4X_2 \)

Subject to
\begin{align*}
5X_1 + 4X_2 & \leq 200 \\
3X_1 + 5X_2 & \leq 150 \\
5X_1 + 4X_2 & \geq 100 \\
8X_1 + 4X_2 & \geq 80
\end{align*}

and \( X_1, X_2 \geq 0 \)
Solution:

Feasible region is given by $OABCD$.

Corner points  

<table>
<thead>
<tr>
<th>Value of $[Z 3X_1, 4X_2]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O (20, 0)$</td>
</tr>
<tr>
<td>$A (40, 0)$</td>
</tr>
<tr>
<td>$B (30.8, 11.5)$</td>
</tr>
<tr>
<td>$C (0, 30)$</td>
</tr>
<tr>
<td>$D (0, 25)$</td>
</tr>
</tbody>
</table>

∴ The maximum value of $Z$ is attained at $B (30.8, 11.5)$.

∴ The optimal solution is $X_1 = 30.8$, $X_2 = 11.5$.

Example 2.13: Use graphical method to solve the LPP.

Maximize $Z = 6X_1 + 4X_2$

Subject to 

$-2X_1 + X_2 \leq 2$

$X_1 - X_2 \leq 2$

$3X_1 + 2X_2 \leq 9$

$X_1, X_2 \geq 0$

Solution:
Feasible region is given by $ABC$.

**Corner points**  
Value of $Z = 3X_1 + 4X_2$

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Value of $Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>(2, 0)</td>
<td>12</td>
</tr>
<tr>
<td>$B$</td>
<td>(3, 0)</td>
<td>18</td>
</tr>
<tr>
<td>$C$</td>
<td>(13/5, 3/5)</td>
<td>98/5 = 19.6 (Maximum value)</td>
</tr>
</tbody>
</table>

The maximum value of $Z$ is attained at $C$ (13/5, 3/5).

$\therefore$ The optimal solution is $X_1 = 13/5, X_2 = 3/5$.

**Example 2.14:** Use graphical method to solve the LPP.

Maximize $Z = 3X_1 + 2X_2$

Subject to

\[
\begin{align*}
5X_1 + X_2 & \geq 10 \\
X_1 + X_2 & \geq 6 \\
X_1 + 4X_2 & \geq 12 \\
x_1, X_2 & \geq 0.
\end{align*}
\]

**Solution:**

<table>
<thead>
<tr>
<th>Corner points</th>
<th>Value of $Z = 3X_1 + 2X_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ (0, 10)</td>
<td>20</td>
</tr>
<tr>
<td>$B$ (1, 5)</td>
<td>13 (Minimum value)</td>
</tr>
<tr>
<td>$C$ (4, 2)</td>
<td>16</td>
</tr>
<tr>
<td>$D$ (12, 0)</td>
<td>36</td>
</tr>
</tbody>
</table>

Since the minimum value is attained at $B$ (1,5) the optimum solution is $X_1, X_2 = 5$.

**Note:** In this problem if the objective function is maximization then the solution is unbounded, as the maximum value of $Z$ occurs at infinity.

**Some more cases**

There are some linear programming problems which may have

(i) A unique optimal solution
(ii) An infinite number of optimal solution
(iii) An unbounded solution
(iv) No solution

The following examples will illustrate these cases.

**Example 2.15:** Solve the LPP by graphical method.
Maximize \( Z = 100X_1 + 40X_2 \)
Subject to 
\[
5X_1 + 2X_2 \leq 1000
\]
\[
3X_1 + 2X_2 \leq 900
\]
\[
X_1 + 2X_2 \leq 500
\]
and \( X_1 + X_2 \geq 0 \)

**Solution:**

The solution space is given by the feasible region OABC.

<table>
<thead>
<tr>
<th>Corner Points</th>
<th>Value of ( Z = 100X_1 + 40X_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>0</td>
</tr>
<tr>
<td>((200, 0))</td>
<td>20,000</td>
</tr>
<tr>
<td>((125, 187.5))</td>
<td>20,000 (Max value of ( Z ))</td>
</tr>
<tr>
<td>((0, 250))</td>
<td>10,000</td>
</tr>
</tbody>
</table>

\( \therefore \) The maximum value of \( Z \) occurs at two vertices \( A \) and \( B \).

Since there are infinite number of points on the line joining \( A \) and \( B \) which gives the same maximum value of \( Z \).

Thus, there are infinite number of optimal solutions for the LPP.

**Example 2.16:** Solve the following LPP.
Max \( Z = 3X_1 + 2X_2 \)
Subject to 
\[
X_1 + X_2 \geq 1
\]
\[
X_1 + X_2 \geq 3
\]
\[
X_1, X_2 \geq 0
\]

**Solution:** The solution space is unbounded. The value of the objective function at the vertices \( A \) and \( B \) are \( Z (A) = 6 \), \( Z (B) = 6 \). But, there exist points in the convex region for which the value of the objective function is more than 8. In
fact, the maximum value of $Z$ occurs at infinity. Hence, the problem has an unbounded

**Solution:**

\[ \text{No feasible solution} \]

When there is no feasible region formed by the constraints in conjunction with non-negativity conditions, then no solution to the LPP exists.

**Example 2.17:** Solve the following LPP.

Max $Z = X_1 + X_2$

Subject to the constraints

\[ X_1 + X_2 \leq 1 \]
\[ -3X_1 + X_2 \geq 3 \]
\[ X_1, X_2 \geq 0 \]

**Solution:**

There being no point $(X_1, X_2)$ common to both the shaded regions, we cannot find a feasible region for this problem. So the problem cannot be solved. Hence, the problem has no solution.

### 2.3.2 General Formulation of LPP

The general formulation of the LPP can be stated as follows: Maximize or Minimize

\[ Z = C_1X_1 + C_2X_2 + ... + C_nX_n \]

...(1)
Subject to \( m \) constraints

\[
\begin{align*}
  a_{11}X_1 + a_{12}X_2 + \cdots + a_{ij}X_j + \cdots + a_{in}X_n & \leq b_1 \\
  a_{21}X_1 + a_{22}X_2 + \cdots + a_{2j}X_j + \cdots + a_{2n}X_n & \leq b_2 \\
  \vdots & \vdots \\
  a_{ij}X_1 + a_{ij}X_2 + \cdots + a_{ij}X_j + \cdots + a_{ijn}X_n & \leq b_j \\
  \vdots & \vdots \\
  a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mj}X_j + \cdots + a_{mn}X_n & \leq b_m
\end{align*}
\]

... (2)

In order to find the values of \( n \) decision variables \( X_1, X_2, \ldots, X_n \) to maximize or minimize the objective function and the non-negativity restrictions

\[ X_1 \geq 0, X_2 \geq 0, \ldots, X_n \geq 0 \] \( \ldots \) (3)

### 2.3.3 Matrix Form of LPP

The linear programming problem can be expressed in the matrix form as follows:

Maximize or Minimize \( Z = CX \)

Subject to \[
\begin{pmatrix}
  \leq \\
  = \\
  \geq
\end{pmatrix}
= b
\]

\( X \geq 0. \)

Where \( x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \), \( b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \), \( C = (C_1C_2 - C_n) \)

and \( A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} \)

### 2.3.4 Some Important Definitions

1. A set of values \( X_1, X_2, \ldots, X_n \) which satisfies the constraints (2) of the LPP is called its solution.

2. Any solution to a LPP which satisfies the non-negativity restrictions (3) of the LPP is called its feasible solution.

3. Any feasible solution which optimizes (Minimizes or Maximizes) the objective function (1) of the LPP is called its optimum solution.
4. Given a system of \( m \) linear equations with \( n \) variables \((m < n)\) any solution which is obtained by solving for \( m \) variables keeping the remaining \( n-m \) variables zero is called a basic solution. Such \( m \) variables are called basic variables and the remaining variables are called non-basic variables.

5. A basic feasible solution is a basic solution which also satisfies (3), that is all basic variables are non-negative.

Basic feasible solutions are of two types:

(a) Non-degenerate: A non-degenerate basic feasible solution is the basic feasible solution which has exactly \( m \) positive \( X_i \) \((i = 1, 2 \ldots m)\) i.e. None of the basic variables are zero.

(b) Degenerate: A basic feasible solution is said to degenerate if one or more basic variables are zero.

6. If the value of the objective function \( Z \) can be increased or decreased indefinitely such solutions are called unbounded solutions.

2.3.5 Canonical or Standard Forms of LPP

The general LPP can be put in the following forms, namely canonical and standard forms.

In the standard form, irrespective of the objective function, namely maximize or minimize, all the constraints are expressed as equations. Moreover, RHS of each constraint and all variables are non-negative.

Characteristics of the Standard Form

(i) The objective function is of maximization type.
(ii) All constraints are expressed as equations.
(iii) The right hand side of each constraint is non-negative.
(iv) All variables are non-negative.

In the canonical form, if the objective function is of maximization, all the constraints other than non-negativity conditions are \( \leq \) type. If the objective function is of minimization, then all the constraints other than non-negative condition are \( \geq \) type.

Characteristics of the Canonical Form

(i) The objective function is of maximization type.
(ii) All constraints are of \( (\leq) \) type.
(iii) All variables \( X_i \) are non-negative.

Note:

(i) Minimization of a function \( Z \) is equivalent to maximization of the negative expression of this function, i.e., \( \text{Min } Z = -\text{Max } (-Z) \).
(ii) An inequality in one direction can be converted into an inequality in the opposite direction by multiplying both sides by \((-1)\).
(iii) Suppose we have the constraint equation

\[
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1
\]
This equation can be replaced by two weak inequalities in opposite directions

\[ a_{11}x_1 + a_{12}x_2 + \ldots + a_{mn}x_n \leq b \]
\[ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b \]

(iv) If a variable is unrestricted in sign, then it can be expressed as a difference of two non-negative variables, i.e., if \( X_i \) is unrestricted in sign, then \( X_i = X_i' - X_i'' \), where \( X_i' \) and \( X_i' - X_i'' \) are \( \geq 0 \).

(v) In the standard form, all the constraints are expressed in equation, which is possible by introducing some additional variables called slack variables and surplus variables so that a system of simultaneous linear equations is obtained. The necessary transformation will be made to ensure that \( b_i \geq 0 \).

**Definition**

(i) If the constraints of a general LPP be

\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, 2, \ldots, m) \]

Then the non-negative variables \( S_i \) which are introduced to convert the inequalities \( (\leq) \) to the equalities

\[ \sum_{j=1}^{n} a_{ij} x_j + S_i = b_i \quad (i = 1, 2, \ldots, m) \]

are called slack variables.

Slack variables are also defined as the non-negative variables which are added in the LHS of the constraint to convert the inequality ‘\( \leq \)’ into an equation.

(ii) If the constraints of a general LPP be

\[ \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad (i = 1, 2, \ldots, m) \]

Then, the non-negative variables \( S_i \) which are introduced to convert the inequalities \( (\geq) \) to the equalities

\[ \sum_{j=1}^{n} a_{ij} x_j - S_i = b_i \quad (i = 1, 2, \ldots, m) \]

are called surplus variables.

Surplus variables are defined as the non-negative variables which are removed from the LHS of the constraint to convert the inequality \( (\geq) \) into an equation.

### 2.4 BIG M METHOD

The following steps are involved in solving an LPP using the Big \( M \) method.

**Step 1** Express the problem in the standard form.

**Step 2** Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type \( \geq \) or \( = \). However, addition of these artificial variable causes violation of the corresponding constraints. Therefore, we would like to get rid of these variables and would not allow them to appear in the final solution. This is achieved by assigning a very large penalty (\( -M \) for maximization and \( M \) for minimization) in the objective function.
Step 3 Solve the modified LPP by the simplex method, until any one of the three cases may arise.

1. If no artificial variable appears in the basis and the optimality conditions are satisfied, then the current solution is an optimal basic feasible solution.

2. If at least one artificial variable in the basis at zero level and the optimality condition is satisfied then the current solution is an optimal basic feasible solution (though degenerated solution).

3. If at least one artificial variable appears in the basis at positive level and the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function, since it contains a very large penalty $M$ and is called pseudo-optimal solution.

Note: While applying the simplex method, whenever an artificial variable happens to leave the basis, we drop that artificial variable and omit all the entries corresponding to its column from the simplex table.

Example 2.18: Use penalty method to

Maximize $Z = 3x_1 + 2x_2$

Subject to the constraints

$2x_1 + x_2 \leq 2$
$3x_1 + 4x_2 \geq 12$
$x_1, x_2 \geq 0$

Solution: By introducing slack variable $S_1 \geq 0$, surplus variable $S_2 \geq 0$ and artificial variable $A_1 \geq 0$, the given LPP can be reformulated as:

Maximize $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA_1$,

Subject to $2x_1 + x_2 + S_1 = 2$
$3x_1 + 4x_2 - S_2 + A_1 = 12$

The starting feasible solution is $S_1 = 2, A_1 = 12$.

Initial table

<table>
<thead>
<tr>
<th>$C_j$</th>
<th>3</th>
<th>2</th>
<th>0</th>
<th>0</th>
<th>$-M$</th>
<th>$Min \ x_j/x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_B$</td>
<td>$B$</td>
<td>$x_B$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$S_1$</td>
<td>$S_2$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$S_1$</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$-M$</td>
<td>12</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Z_j$</td>
<td>$-12M$</td>
<td>$-3M$</td>
<td>$-4M$</td>
<td>0</td>
<td>$M$</td>
<td>$-M$</td>
</tr>
<tr>
<td>$Z_j - C_j$</td>
<td>$-3M - 3$</td>
<td>$-4M - 2$</td>
<td>$0$</td>
<td>$M$</td>
<td>$0$</td>
<td></td>
</tr>
</tbody>
</table>

Since some of the $Z_j - C_j \leq 0$, the current feasible solution is not optimum. Choose the most negative $Z_j - C_j = -4M - 2$.

$\therefore x_2$ variable enters the basis, and the basic variable $S_1$ leaves the basis.
Since all \( Z_j - C_j \geq 0 \) and an artificial variable appears in the basis at positive level, the given LPP does not possess any feasible solution. But the LPP possesses a pseudo-optimal solution.

**Example 2.19:** Solve the LPP

Minimize \( Z = 4x_1 + x_2 \)

Subject to

\[
\begin{align*}
3x_1 + x_2 &= 3 \\
4x_1 + 3x_2 &\geq 6 \\
x_1 + 2x_2 &\leq 3 \\
x_1, x_2 &\geq 0
\end{align*}
\]

**Solution:** Since the objective function is minimization, we convert it into maximization using

Min \( Z = -\text{Max} (-z) \)

Maximize \( Z = -4x_1 - x_2 \)

Subject to

\[
\begin{align*}
3x_1 + x_2 &= 3 \\
4x_1 + 3x_2 &\geq 6 \\
x_1 + 2x_2 &\leq 3 \\
x_1, x_2 &\geq 0
\end{align*}
\]

Convert the given LPP into the standard form by adding artificial variables \( A_1, A_2 \), surplus variable \( S_1 \) and slack variable \( S_2 \) to get the initial basic feasible solution.

Maximize \( Z = -4x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2 \)

Subject to

\[
\begin{align*}
3x_1 + x_2 + A_1 &= 3 \\
4x_1 + 3x_2 - S_1 + A_2 &= 6 \\
x_1 + 2x_2 + S_2 &= 3
\end{align*}
\]

The starting feasible solution is \( A_1 = 3, A_2 = 6, S_2 = 3 \).
### Initial solution

<table>
<thead>
<tr>
<th>$C_j$</th>
<th>$-4$</th>
<th>$-1$</th>
<th>$-M$</th>
<th>$0$</th>
<th>$-M$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_B$</td>
<td>$B$</td>
<td>$x_B$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$A_1$</td>
<td>$S_1$</td>
</tr>
<tr>
<td>$-M$</td>
<td>$A_1$</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$-M$</td>
<td>$A_2$</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\leftarrow$</td>
<td>$S_2$</td>
<td>3</td>
<td>1</td>
<td>$\underline{2}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$Z_j$</th>
<th>$-9M$</th>
<th>$-7M$</th>
<th>$-4M$</th>
<th>$-M$</th>
<th>$-M$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_j - C_j$</td>
<td>$-7M + 4$</td>
<td>$-4M + 1\uparrow$</td>
<td>$0$</td>
<td>$M$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Since some of the $Z_j - C_j \geq 0$, the current feasible solution is not optimum. As $Z_1 - C_1$ is most negative, $x_1$ enters the basis and the basic variable $A_2$ leaves the basis.

### First iteration

<table>
<thead>
<tr>
<th>$C_j$</th>
<th>$-4$</th>
<th>$-1$</th>
<th>$-M$</th>
<th>$0$</th>
<th>$-M$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_B$</td>
<td>$B$</td>
<td>$x_B$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$A_1$</td>
<td>$S_1$</td>
</tr>
<tr>
<td>$-M$</td>
<td>$A_1$</td>
<td>$3/2$</td>
<td>$5/2$</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\leftarrow -M$</td>
<td>$A_2$</td>
<td>$3/2$</td>
<td>$\underline{5/2}$</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$x_2$</td>
<td>$3/2$</td>
<td>$3/2$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$Z_j$</th>
<th>$-3M - 3/2$</th>
<th>$-5M - 1/2$</th>
<th>$-1$</th>
<th>$-M$</th>
<th>$+M$</th>
<th>$-M$</th>
<th>$2M - 1/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_j - C_j$</td>
<td>$-5M - 7/2\uparrow$</td>
<td>$0$</td>
<td>$0$</td>
<td>$M$</td>
<td>$0$</td>
<td>$2M - 1/2$</td>
<td></td>
</tr>
</tbody>
</table>

Since $Z_1 - C_1$ is negative, the current feasible solution is not optimum. Therefore, $x_1$ variable enters the basis and the artificial variable $A_2$ leaves the basis.

### Second iteration

<table>
<thead>
<tr>
<th>$C_j$</th>
<th>$-4$</th>
<th>$-1$</th>
<th>$-M$</th>
<th>$0$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_B$</td>
<td>$B$</td>
<td>$x_B$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>$-M$</td>
<td>$A_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\leftarrow -M$</td>
<td>$x_1$</td>
<td>$\frac{3}{5}$</td>
<td>$1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$-1$</td>
<td>$x_2$</td>
<td>$\frac{6}{5}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$Z_j$</th>
<th>$\frac{18}{5}$</th>
<th>$-4$</th>
<th>$-1$</th>
<th>$-M$</th>
<th>$-M+\frac{9}{5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_j - C_j$</td>
<td>$\frac{9}{5}\uparrow$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-M\frac{9}{5}$</td>
</tr>
</tbody>
</table>
Since $Z_4 - C_4$ is most negative, $S_1$ enters the basis and the artificial variable $A_1$ leaves the basis.

**Third iteration**

<table>
<thead>
<tr>
<th>$C_j$</th>
<th>$-4$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_B$</td>
<td>$B$</td>
<td>$x_B$</td>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>← 0</td>
<td>$S_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$-4$</td>
<td>$x_1$</td>
<td>$\frac{3}{5}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$-1$</td>
<td>$x_2$</td>
<td>$\frac{6}{5}$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$Z_j$</td>
<td>18</td>
<td>$-4$</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>$Z_j - C_j$</td>
<td>$\frac{5}{5}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Since $Z_4 - C_4$ is most negative, $S_2$ enters the basis and $S_1$ leaves the basis.

**Fourth iteration**

<table>
<thead>
<tr>
<th>$C_j$</th>
<th>$-4$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_B$</td>
<td>$B$</td>
<td>$x_B$</td>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>0</td>
<td>$S_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$-4$</td>
<td>$x_1$</td>
<td>$\frac{3}{5}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$-1$</td>
<td>$x_2$</td>
<td>$\frac{6}{5}$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$Z_j$</td>
<td>$-\frac{18}{5}$</td>
<td>$-4$</td>
<td>$-1$</td>
<td>$1/5$</td>
</tr>
<tr>
<td>$Z_j - C_j$</td>
<td>0</td>
<td>0</td>
<td>$1/5$</td>
<td>0</td>
</tr>
</tbody>
</table>

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by $x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$, and Max $Z = -18/5$.

∴ $Min Z = -\text{Max} (-Z) = 18/5$.

**Example 2.20:** Solve by Big $M$ method.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to $x_1 + 2x_2 + 3x_3 = 15$
$2x_1 + x_2 + 5x_3 = 20$
$x_1 + 2x_2 + x_3 + x_4 = 10$

**Solution:** Since the constraints are equations, introduce artificial variables $A_1$, $A_2 \geq 0$. The reformulated problem is given as follows.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

Subject to $x_1 + 2x_2 + 3x_3 + A_1 = 15$
2x₁ + x₂ + 5x₃ + A₁ = 20
x₁ + 2x₂ + x₃ + x₄ = 10

Initial solution is given by A₁ = 15, A₂ = 20 and x₄ = 10.

Initial table

<table>
<thead>
<tr>
<th>Cj</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>–1</th>
<th>–M</th>
<th>–M</th>
<th>Min ( \frac{xB}{x₃} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cᵢₙ</td>
<td>B</td>
<td>x₁</td>
<td>x₂</td>
<td>x₃</td>
<td>x₄</td>
<td>A₁</td>
<td>A₂</td>
</tr>
<tr>
<td>–M</td>
<td>A₁</td>
<td>15</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>← –M</td>
<td>A₂</td>
<td>20</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>–1</td>
<td>x₄</td>
<td>10</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Zᵢ = \( Z_j – C_j \)

Since \( Z_j – C_j \) is most negative, \( x₃ \) enters the basis and the basic variable \( A₂ \) leaves the basis.

First iteration

<table>
<thead>
<tr>
<th>Cj</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>–1</th>
<th>–M</th>
<th>–M</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cᵢₙ</td>
<td>B</td>
<td>x₁</td>
<td>x₂</td>
<td>x₃</td>
<td>x₄</td>
<td>A₁</td>
</tr>
<tr>
<td>← –M</td>
<td>A₁</td>
<td>3</td>
<td>–1/5</td>
<td>7/5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>A₂</td>
<td>4</td>
<td>2/5</td>
<td>1/5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>–1</td>
<td>x₄</td>
<td>6</td>
<td>3/5</td>
<td>9/5</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Zᵢ = \( Z_j – C_j \)

Since \( Z_j – C_j \) is most negative, \( x₂ \) enters the basis and the basic variable \( A₁ \) leaves the basis.
Second iteration

<table>
<thead>
<tr>
<th></th>
<th>$C_j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>–1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_B$</td>
<td>$B$</td>
<td>$x_B$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
</tr>
<tr>
<td>2</td>
<td>$x_2$</td>
<td>15/7</td>
<td>( \frac{1}{7} )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$x_3$</td>
<td>25/7</td>
<td>3/7</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \leftarrow -1 )</td>
<td>$x_4$</td>
<td>15/7</td>
<td>( \frac{6}{7} )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$Z_j$ = \frac{90}{7} \text{ \( \frac{1}{7} \text{ } – \text{ } 6/7 \uparrow \)} 2 3 \(-1\)
$Z_j - C_j$ = 0 0 0

Since $Z_j - C_j = -6/7$ is negative, the current feasible solution is not optimum. Therefore, $x_1$ enters the basis and the basic variable $x_4$ leaves the basis.

Since all $Z_j - C_j \geq 0$, the solution is optimum and is given by $x_1 = x_2 = x_3 = 15/6 = 5/2$, and Max $Z = 15$.

2.5 SIMPLEX METHOD

2.5.1 Introduction

Simplex method is an iterative procedure for solving LPP in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be than at the previous vertex. This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of unbounded solution.

2.5.2 Definition

(i) Let $X_B$ be a basic feasible solution to the LPP.
Max $Z = C_X$
Subject to $A_X = b$
and \( X \geq 0 \), such that it satisfies \( X_B = B^{-1}b \)

where \( B \) is the basis matrix formed by the column of basic variables.

The vector \( C_B = (C_B^1, C_B^2, \ldots, C_B^m) \) where \( C_B^j \) are components of \( C \) associated with the basic variables is called the cost vector associated with the basic feasible solution \( X_B^r \).

(ii) Let \( X_B^r \) be a basic feasible solution to the LPP.

Max \( Z = C_X \) where
\[ A_X = b \] and \( X \geq 0 \).

Let \( C_B \) be the cost vector corresponding to \( X_B^r \). For each column vector \( a_j \) in \( A \), which is not a column vector of \( B \), let
\[ a_j = a_j^i b_i \]

Then the number \( Z_j = \sum_{i=1}^{m} C_B^i a_{ij} \)

is called the evaluation corresponding to \( a_j \) and the number \((Z_j - C_j)\) is called the net evaluation corresponding to \( j \).

**Simplex algorithm**

For the solution of any LPP by simplex algorithm the existence of an initial basic feasible solution is always assumed. The steps for the computation of an optimum solution are as follows:

**Step 1** Check whether the objective function of the given LPP is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximization by

\[ \text{Min } Z = -\text{Max } (-Z) \]

**Step 2** Check whether all \( b_i \) \((i = 1, 2, \ldots, m)\) are positive. If any one of \( b_i \) is negative then multiply the inequation of the constraint by \(-1\) so as to get all \( b_i \) to be positive.

**Step 3** Express the problem in standard form by introducing slack/surplus variables, to convert the inequality constraints into equations.

**Step 4** Obtain an initial basic feasible solution to the problem in the form \( X_B^r = B^{-1}b \) and put it in the first column of the simplex table. Form the initial simplex table as follows:

<table>
<thead>
<tr>
<th>( C_B )</th>
<th>( S_B )</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( \ldots )</th>
<th>( \ldots )</th>
<th>( 0 )</th>
<th>( 0 )</th>
<th>( \ldots )</th>
<th>( 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{g1} )</td>
<td>( S_1 )</td>
<td>( b_1 )</td>
<td>( a_{11} )</td>
<td>( a_{12} )</td>
<td>( a_{13} )</td>
<td>( a_{14} )</td>
<td>( \ldots )</td>
<td>( a_{1m} )</td>
<td>( 1 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( C_{g2} )</td>
<td>( S_2 )</td>
<td>( b_2 )</td>
<td>( a_{21} )</td>
<td>( a_{22} )</td>
<td>( a_{23} )</td>
<td>( a_{24} )</td>
<td>( \ldots )</td>
<td>( a_{2m} )</td>
<td>( 1 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>
Step 5 Compute the net evaluations $Z_j - C_j$ by using the relation

$$Z_j - C_j = C_b \cdot (a_j - c).$$

Examine the sign of $Z_j - C_j$.

(i) If all $Z_j - C_j \geq 0$, then the initial basic feasible solution $X_B$ is an optimum basic feasible solution.

(ii) If at least one $Z_j - C_j > 0$, then proceed to the next step as the solution is not optimal.

Step 6 (To find the entering variable i.e., key column)

If there are more than one negative $Z_j - C_j$, choose the most negative of them. Let it be $Z_j - C_j$ for some $j = r$. This gives the entering variable $X_r$ and is indicated by an arrow at the bottom of the $r^{th}$ column. If there are more than one variables having the same most negative $Z_j - C_j$ then any one of the variable can be selected arbitrarily as the entering variable.

(i) If all $X_{ir} \leq 0$ ($i = 1, 2 \ldots m$) then there is an unbounded solution to the given problem.

(ii) If at least one $X_{ir} > 0$ ($i = 1, 2 \ldots m$) then the corresponding vector $X_r$ enters the basis.

Step 7 (To find the leaving variable or key row)

Compute the ratio ($X_{Bi}/X_{ir}, X_{ir} > 0$).

If the minimum of these ratios be $X_{Bi}/X_{ir}$, then choose the variable $X_i$ to leave the basis called the key row and the element at the intersection of key row and key column is called the key element.

Step 8 Form a new basis by dropping the leaving variable and introducing the entering variable along with the associated value under $C_B$ column. Convert the leading element to unity by dividing the key equation by the key element and all other elements in its column to zero by using the formula

$$\text{New Element} = \frac{\text{Old Element} - \text{Product of elements in key row and key column}}{\text{key element}}$$

Step 9 Go to step (5) and repeat the procedure until either an optimum solution is obtained or there is an indication of unbounded solution.

Example 2.22: Use simplex method to solve the LPP.

Max $Z = 3X_1 + 2X_2$

Subject to

$$\begin{align*}
X_1 + X_2 & \leq 4 \\
X_1 + X_2 & = 2 \\
X_1, X_2 & \geq 0
\end{align*}$$

Solution: By introducing the slack variables $S_1, S_2$, convert the problem into the standard form.

Max $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2$
Subject to \[ \begin{align*} X_1 & \quad X_2 & \quad S_1 & \quad 4 \\ X_1 & \quad X_2 & \quad S_2 & \quad 2 \\ X_1, X_2, S_1, S_2 & \quad 0 \end{align*} \]

NOTES

\[ \begin{array}{cccc|c|c} X_1 & X_2 & S_1 & S_2 & X_1 & 4 \\ 1 & 1 & 1 & 0 & S_1 & 2 \\ 1 & 1 & 0 & 1 & S_2 & \end{array} \]

An initial basic feasible solution is given by \[ X_B = B^{-1}b, \] where \[ B = I_2, X_B = (S_1, S_2). \]
i.e., \[ (S_1, S_2) = I_2 (4,2) = (4,2) \]

**Initial simplex table**

\[ Z_j = C_B a_j \]

\[ Z_1 - c_1 = C_B a_1 - c_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3 = -3 \]

\[ Z_2 - c_2 = C_B a_2 - c_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1-1) - 2 = -2 \]

\[ Z_3 - c_3 = C_B a_3 - c_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1 0) - 0 = 0 \]

\[ Z_4 - c_4 = C_B a_4 - c_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (0 1) - 0 = 0 \]

\[
\begin{array}{cccc|c|c|c}
    & C_j & 3 & 2 & 0 & 0 & Min \frac{X_B}{X_1} \\
    \hline
    C_B & B & X_B & X_1 & X_2 & S_1 & S_2 \\
    \hline
    0 & S_1 & 4 & 1 & 1 & 1 & 0 & 4/1 = 4 \\
    \leftarrow 0 & S_2 & 2 & 1 & -1 & 0 & 1 & 2/1 = 2 \\
    \hline
    Z_j & 0 & 0 & 0 & 0 & 0 & 0 \\
    Z_j - C_j & -3 \uparrow & -2 & 0 & 0 & 0 & \\
\end{array}
\]

Since there are some \( Z_j - C_j = 0 \), the current basic feasible solution is not optimum.

Since \( Z_1 - C_1 = -3 \) is the most negative, the corresponding non-basic variable \( X_1 \) enters the basis.
The column corresponding to this $X_i$ is called the key column.

To find the ratio $= \text{Min} \frac{X_{bi}}{X_{ir}} \times 0$

$= \text{Min} \frac{4 \times 2}{2 \times 1} = 2$ which corresponds to $S_2$.

∴ The leaving variable is the basic variable $S_2$. This row is called the key row.

Convert the leading element $X_{21}$ to units and all other elements in its column, i.e., $(X_i)$ to zero by using the formula:

New element = Old element –

Product of elements in key row and key column

key element

To apply this formula, first we find the ratio, namely

the element to be zero \[ \frac{1}{1} \]

key element \[ \frac{1}{1} \]

Apply this ratio, for the number of elements that are converted in the key row. Multiply this ratio by key row element shown as follows.

1 × 2
1 × 1
1 × −1
1 × 0
1 × 1

Now, subtract this element from the old element. The elements to be converted into zero, is called the old element row. Finally we have

4 − 1 × 2 = 2
1 − 1 × 1 = 0
1 − 1 × −1 = 2
1 − 1 × 0 = 1
0 − 1 × 1 = −1

∴ The improved basic feasible solution is given in the following simplex table
First iteration

<table>
<thead>
<tr>
<th>C_b</th>
<th>B</th>
<th>x_b</th>
<th>x_1</th>
<th>x_2</th>
<th>S_1</th>
<th>S_2</th>
<th>S_3</th>
<th>Min X_b X_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>S_1</td>
<td>12</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>12/4 = 4</td>
</tr>
<tr>
<td>0</td>
<td>S_2</td>
<td>8</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>8/4 = 2</td>
</tr>
<tr>
<td>←0</td>
<td>S_3</td>
<td>8</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>8/4 = 2</td>
</tr>
</tbody>
</table>

Z_j   0   0   0   0   0   0
Z_j - C_j -3↑ -2   0   0   0

Since Z_2 - C_2 is most negative, X_2 enters the basis.

To find Min \( \frac{X_b}{X_2} \) = 0

\[
\text{Min} \frac{2}{2}, \quad -1.
\]

This gives the outgoing variables. Convert the leading element into 1. This is done by dividing all the elements in the key row by 2. The remaining element is multiplied by zero using the formula as shown below.

\(-\frac{1}{2}\) is the common ratio. Put this ratio 5 times and multiply each ratio by key row element.

\[
\begin{align*}
\frac{1}{2} & \quad 2 \\
\frac{1}{2} & \quad 0 \\
\frac{1}{2} & \quad 2 \\
-1/2 \times 2 & \\
-1/2 \times -1 & \\
\end{align*}
\]

Subtract this from the old element. All the row elements which are converted into zero are called the old elements.

\[
\begin{align*}
2 & \quad \frac{1}{2} \quad 2 \quad 3 \\
1 - (-1/2 \times 0) & = 1 \\
-1 - (-1/2 \times 2) & = 1 \\
0 - (-1/2 \times 1) & = 1/2 \\
1 - (-1/2 \times 1) & = 1/2 \\
\end{align*}
\]
Second iteration

\[
\begin{array}{c|cccccc}
C_j & 1 & 1 & 3 & 0 & 0 \\
\hline
C_B & B & X_B & X_1 & X_2 & S_3 & S_1 & S_2 \\
0 & S_1 & 2 & 2 & 3/2 & 0 & 1 & -1/2 \\
3 & X_1 & 1 & 1 & 1/2 & 1 & 0 & 1/2 \\
\hline
Z_j & 3 & 3 & 3/2 & 3 & 0 & 3/2 \\
Z_j - C_j & 2 & 1/2 & 0 & 0 & 3/2 \\
\end{array}
\]

Since all \( Z_j - C_j \geq 0 \), the solution is optimum. The optimal solution is \( \text{Max } Z = 11 \), \( X_1 = 3 \), and \( X_2 = 1 \).

**Example 2.23:** Solve the LPP

\[
\text{Max } Z = 3X_1 + 2X_2 \\
\text{Subject to } \begin{align*}
4X_1 + 3X_2 & \leq 12 \\
4X_1 + X_2 & \leq 8 \\
4X_1 + X_2 & \leq 8 \\
X_1, X_2 & \geq 0
\end{align*}
\]

**Solution:** Convert the inequality of the constraint into an equation by adding slack variables \( S_1, S_2, S_3 \).

\[
\text{Max } Z = 3X_1 + 2X_2 + 0S_1 + 0S_2 + 0S_3 \\
\text{Subject to } \begin{align*}
4X_1 + 3X_2 + S_1 & = 12 \\
4X_1 + X_2 + S_2 & = 8 \\
4X_1 + X_2 + S_3 & = 8 \\
X_1, X_2, S_1, S_2, S_3 & \geq 0
\end{align*}
\]
### Initial table

<table>
<thead>
<tr>
<th>(C_B)</th>
<th>(B)</th>
<th>(X_B)</th>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(S_1)</th>
<th>(S_2)</th>
<th>(S_3)</th>
<th>(Min \frac{X_B}{X_1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(S_1)</td>
<td>12</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>(12/4 = 3)</td>
</tr>
<tr>
<td>0</td>
<td>(S_2)</td>
<td>8</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>(8/4 = 2)</td>
</tr>
<tr>
<td>(-0)</td>
<td>(S_3)</td>
<td>8</td>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>(8/4 = 2)</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
Z_j &= 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
Z_j - C_j &= -3 \uparrow \quad -2 \quad 0 \quad 0 \quad 0
\end{align*}
\]

\[
\therefore Z_j - C_j \quad \text{is most negative, } X_1 \quad \text{enters the basis. And the min } \left( \frac{X_B}{X_1}, X_{il} > 0 \right) = \min(3, 2, 2) = 2 \text{ gives } S_j \text{ as the leaving variable.}
\]

Convert the leading element into 1, by dividing key row element by 4 and the remaining elements into 0.

### First iteration

<table>
<thead>
<tr>
<th>(C_J)</th>
<th>3</th>
<th>2</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>(C_B)</th>
<th>(B)</th>
<th>(X_B)</th>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(S_1)</th>
<th>(S_2)</th>
<th>(S_3)</th>
<th>(Min \frac{X_B}{X_2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(S_1)</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>(4/4 = 1)</td>
</tr>
<tr>
<td>(-0)</td>
<td>(S_2)</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>(0/2 = 1)</td>
</tr>
<tr>
<td>3</td>
<td>(X_1)</td>
<td>2</td>
<td>1</td>
<td>-1/4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(1/4)</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
Z_j &= 6 \quad 3 \quad -3/4 \quad 0 \quad 0 \quad 3/4 \\
Z_j - C_j &= 0 \quad -11/4 \uparrow \quad 0 \quad 0 \quad 3/4
\end{align*}
\]

\[
\begin{align*}
8 & \quad 4 \quad 8 \quad 0 \quad 12 & \quad 4 \quad 8 \quad 4 \\
4 & \quad 4 \quad 4 \quad 0 & \quad 4 & \quad 4 \quad 4 \quad 0 \\
1 & \quad 4 \quad 1 \quad 2 & \quad 3 & \quad 4 \quad 1 \quad 4 \\
0 & \quad 4 \quad 0 \quad 0 & \quad 4 & \quad 0 \quad 1
\end{align*}
\]


Since $Z_2 - C_2 = 3/4$ is the most negative, $x_2$ enters the basis.

To find the outgoing variable, find $\min \frac{x_B}{x_i}$, $x_{i2} = 0$

$\min \left( \frac{4}{2} - 1 \right) = 0$

**First iteration**

Therefore $S_1$ leaves the basis. Convert the leading element into 1 by dividing the key row elements by 2 and remaining element in that column into zero using the formula.

New element = old element

<table>
<thead>
<tr>
<th>Product of elements in key row and key column</th>
</tr>
</thead>
<tbody>
<tr>
<td>key element</td>
</tr>
<tr>
<td>$C_1$ 3 2 0 0 0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$C_3$</th>
<th>$B$</th>
<th>$X_3$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_1$</th>
<th>$\text{Min} \frac{X_3}{S_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>←0</td>
<td>$S_1$</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>−2</td>
<td></td>
<td>4/1 = 1</td>
</tr>
<tr>
<td>2</td>
<td>$X_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>½</td>
<td>−½</td>
<td>−</td>
</tr>
<tr>
<td>3</td>
<td>$X_1$</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1/8</td>
<td>1/8</td>
<td>2/1/8 = 16</td>
</tr>
<tr>
<td></td>
<td>$Z_j$</td>
<td>6</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>11/8</td>
<td>−5/8</td>
<td>−</td>
</tr>
<tr>
<td></td>
<td>$Z_j - C_j$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>11/8</td>
<td>−5/8</td>
<td>−</td>
</tr>
</tbody>
</table>

**Second iteration**

Since $Z_j - C_j = −5/8$ is most negative, $S_3$ enters the basis and

$\text{Min} \frac{X_B}{S_{13}}$, $S_{13}$  $\text{Min} \frac{4}{1}$, $\frac{1 - 2}{1/18}$  4

Therefore, $S_1$ leaves the basis. Convert the leading element into one and the remaining elements into zero.
Third iteration

<table>
<thead>
<tr>
<th>$C_B$</th>
<th>$B$</th>
<th>$X_B$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$S_1$</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$S_2$</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$X_1$</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$Z_i$</td>
<td>17/2</td>
<td>3</td>
<td>2</td>
<td>5/8</td>
<td>1/8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$Z_j - C_j$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5/8</td>
<td>1/8</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Since all $Z_i - C_j \geq 0$, the solution is optimum and it is given by $X_1 = 3/2, X_2 = 2$ and Max $Z = 17/2$.

**Example 2.24:** Using simplex method solve the LPP.

Maximize $Z = X_1 + X_2 + 3X_3$

Subject to $3X_1 \leq 2X_2 + X_3 \leq 3$
$2X_1 \leq X_2 + 2X_3 \leq 2$
$X_1, X_2, X_3 \geq 0$

**Solution:** Rewrite the inequality of the constraints into an equation by adding slack variables.

Max $Z = X_1 + X_2 + 3X_3 + 0S_1 + 0S_2$

Subject to $3X_1 \leq 2X_2 + X_3 \leq 3$
$2X_1 \leq X_2 + 2X_3 \leq 2$

Initial basic feasible solution is

$X_1, X_2, X_3, S_1, S_2$

$3, 2, 1, 1, 0$

$2, 1, 2, 0, 1$

$1, 1, 3, 0, 0$

<table>
<thead>
<tr>
<th>$C_j$</th>
<th>3</th>
<th>2</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_B$</td>
<td>$B$</td>
<td>$X_B$</td>
<td>$X_1$</td>
<td>$X_2$</td>
<td>$X_3$</td>
<td>$S_1$</td>
</tr>
<tr>
<td>0</td>
<td>$S_1$</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>←0</td>
<td>$S_2$</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$Z_i$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$Z_j - C_j$</td>
<td>-1</td>
<td>-1</td>
<td>-3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Since $Z_3 - C_3 = -3$ is the most negative, the variable $X_3$ enters the basis. The column corresponding to $X_3$ is called the key column.

(To determine the key row or the leaving variable, find $\min \frac{x_{B}}{x_{13}}$, $x_{13}$ = 0 Min

\[
\begin{array}{cc|cc|cc|cc|cc}
3 & 1 & \frac{2}{2} & 1 & 1 \\
\end{array}
\]

Therefore, the leaving variable is the basic variable $S_2$, the row is called the key row and the intersection element 2 is called the key element.

Convert this element into 1 by dividing each element in the key row by 2 and the remaining elements in that key column into zero using the formula

New element = old element –

Product of elements in key row and key column

key element

**First iteration**

<table>
<thead>
<tr>
<th>$C_j$</th>
<th>1</th>
<th>1</th>
<th>3</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_B$</td>
<td>$B$</td>
<td>$X_B$</td>
<td>$X_1$</td>
<td>$X_2$</td>
<td>$X_3$</td>
</tr>
<tr>
<td>0</td>
<td>$S_1$</td>
<td>2</td>
<td>2</td>
<td>3/2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$X_3$</td>
<td>1</td>
<td>1</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>$Z_j$</td>
<td>3</td>
<td>3</td>
<td>3/2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$Z_j - C_j$</td>
<td>2</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>3/2</td>
</tr>
</tbody>
</table>

Since all $Z_j - C_j \geq 0$, the solution is optimum and it is given by $X_1 = 0$, $X_2 = 0$, $X_3 = 1$, $\text{Max } Z = \frac{3}{2}$.

2.6 **DUALITY IN LP: CONVERSION OF PRIMAL TO DUAL**

Every LPP (called the primal) is associated with another LPP (called its dual). Either of the problem can be considered as primal with the other as dual.

The importance of the duality concept is due to two main reasons:

(i) If the primal contains a large number of constraints and a smaller number of variables, the labour of computation can be considerably reduced by converting it in to the dual problem and then solving it. (ii) The interpretation of the dual variables from the cost or economic point of view, proves extremely useful in making future decisions in the activities being programmed.
2.6.1 Formulation of Dual Problem

For formulating a dual problem, first we bring the problem in the canonical form. The following changes are used in formulating the dual problem.

1. Change the objective function of maximization in the primal into that of minimization in the dual and vice versa.
2. The number of variables in the primal will be the number of constraints in the dual and vice versa.
3. The cost coefficients $C_1, C_2, \ldots, C_n$ in the objective function of the primal will be the RHS constant of the constraints in the dual and vice versa.
4. In forming the constraints for the dual, we consider the transpose of the body matrix of the primal problem.
5. The variables in both problems are non-negative.
6. If the variable in the primal is unrestricted in sign, then the corresponding constraint in the dual will be an equation and vice versa.

2.6.2 Definition of Dual Problem

Let the primal problem be

Max $Z = C_1 x_1 + C_2 x_2 + \ldots + C_n x_n$

Subject to

\[
\begin{align*}
\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{array}
\begin{array}{c}x_1 \\
x_2 \\
\vdots \\
x_n
\end{array}
\begin{array}{c}b_1 \\
b_2 \\
\vdots \\
b_m
\end{array}
\end{align*}
\]

Dual: The dual problem is defined as

Min $Z = b_1 w_1 + b_2 w_2 + \ldots + b_m w_m$

Subject to

\[
\begin{align*}
\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1m} \\
a_{21} & a_{22} & \cdots & a_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mm}
\end{array}
\begin{array}{c}w_1 \\
w_2 \\
\vdots \\
w_m
\end{array}
\begin{array}{c}C_1 \\
C_2 \\
\vdots \\
C_m
\end{array}
\end{align*}
\]

where $w_1, w_2, w_3, \ldots, w_m$ are called dual variables.

Example 2.25: Write the dual of the following primal LP problem.

Max $Z = x_1 + 2x_2 + x_3$
Solution: Since the problem is not in the canonical form, we interchange the inequality of the second constraint.

Max $Z = x_1 + 2x_2 + x_3$

Subject to

\begin{align*}
2x_1 & \leq x_2 + x_3 \\
2x_1 & \leq 5x_2 + 6x_3 \\
4x_1 & \leq x_2 + 6x_3 \\
& \geq x_1, x_2, x_3 \\
\end{align*}

Dual: Let $w_1, w_2, w_3$ be the dual variables.

Min $Z^d = 2w_1 + 6w_2 + 6w_3$

Subject to

\begin{align*}
2w_1 & \leq 2w_2 + 4w_3 \\
& \leq w_1, w_2, w_3 \\
5w_1 & \leq 2w_2 + w_3 \\
& \geq w_1, w_2, w_3 \\
\end{align*}

Example 2.26: Find the dual of the following LPP.

Max $Z = 3x_1 - x_2 + x_3$

Subject to

\begin{align*}
4x_1 & \leq 8 \\
8x_1 & \leq 3x_2 + 12 \\
5x_1 & \leq 6x_3 + 13 \\
& \geq x_1, x_2, x_3 \\
\end{align*}

Solution: Since the problem is not in the canonical form, we interchange the inequality of the second constraint.

Max $Z = 3x_1 - x_2 + x_3$

Subject to

\begin{align*}
4x_1 & \leq 0x_3 + 8 \\
8x_1 & \leq 3x_2 + 12 \\
5x_1 & \leq 0x_2 + 6x_3 + 13 \\
& \geq x_1, x_2, x_3 \\
\end{align*}

Max $Z = Cx$

Subject to

\begin{align*}
Ax & \geq B \\
x & \geq 0 \\
\end{align*}
\[ C = (3-11) \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad b = \begin{pmatrix} 8 \\ -12 \\ 13 \end{pmatrix} \]

\[ A = \begin{pmatrix} 4 & -1 & 0 \\ -8 & -1 & -3 \\ 5 & 0 & -6 \end{pmatrix} \]

**Dual:** Let \( w_1, w_2, w_3 \) be the dual variables. The dual problem is

Min \( Z^1 = b^T W \)

Subject to \( A^T W \geq C^T \) and \( W \geq 0 \)

i.e. Min \( Z^1 = (8 \quad -12 \quad 13) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \)

Subject to \( \begin{pmatrix} 4 & -8 & 5 \\ -1 & -1 & 0 \\ 0 & -3 & 6 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \geq \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \)

Min \( Z_1 = 8w_1 - 12w_2 + 13w_3 \)

Subject to \( 4w_1 - 8w_2 + 5w_3 \geq 3 \)
\(-w_1 - w_2 + 0w_3 \geq -1 \)
\(0w_1 - 3w_2 + 6w_3 \geq 1 \)
\(w_1, w_2, w_3 \geq 0 \)

**Example 2.27:** Write the dual of the following LPP

Min \( Z = 2x_2 + 5x_3 \)

Subject to \( \begin{array}{ccc} x_1 & x_2 & 2 \\ 2x_1 & x_2 & 6x_3 & 6 \\ x_1 & x_2 & 3x_3 & 4 \\ x_1, x_2, x_3 & 0 \end{array} \)

**Solution:** Since the given primal problem is not in the canonical form, we interchange the inequality of the constraint. Also, the third constraint is an equation. This equation can be converted into two inequations.

Min \( Z = 0x_1 + 2x_2 + 5x_3 \)
Again on rearranging the constraint, we have
\[ Z = 0x_1 + 2x_2 + 5x_3 \]
Subject to
\[
\begin{align*}
  x_1 & \leq 2 \\
  2x_1 & \leq 6 \\
  x_1 & \leq 3 \\
  x_1 & \leq 3 \\
  x_1, x_2, x_3 & \geq 0
\end{align*}
\]

**Dual:** Since there are four constraints in the primal, we have four dual variables, namely \( w_1, w_2, w_3', w_3'' \).
Max \( Z' = 2w_1 - 6w_2 + 4w_3' - 4w_3'' \)
Subject to
\[
\begin{align*}
  w_1 & \leq 2 \\
  w_1 & \leq 2 \\
  w_1 & \leq 6 \\
  w_1 & \leq 6 \\
  w_1, w_2, w_3', w_3'' & \geq 0
\end{align*}
\]
Let \( w_3 \ w_3 \ w_3 \)
Max \( Z = 2w_1 + 6w_2 + 4(w_3 \ w_3) \)
Subject to
\[
\begin{align*}
  w_1 & \leq 2 \\
  w_1 & \leq 2 \\
  w_1 & \leq 6 \\
  w_1 & \leq 6 \\
  w_1, w_2, w_3', w_3'' & \geq 0
\end{align*}
\]
Finally, we have, \( 0w_1 \ 6w_2 \ 3(w_3 \ w_3) \ 5 \)
Max \( Z = 2w_1 + 6w_2 + 4w_3 \)
Subject to
\[
\begin{align*}
  w_1 & \leq 2 \\
  w_1 & \leq 2 \\
  w_1 & \leq 6 \\
  w_1 & \leq 6 \\
  w_1, w_2, w_3' \ w_3'' \ & \geq 0
\end{align*}
\]
\( w_1 \ w_2 \ w_3 \) is unrestricted.

**Example 2.28:** Give the dual of the following problem
Max \( Z = x + 2y \)
Subject to \(2x \leq 3y \leq 4\)
\(3x \leq 4y \leq 5\)
\(x \geq 0\) and \(y\) unrestricted.

**Solution:** Since the variable \(y\) is unrestricted, it can be expressed as \(y \geq 0\). On reformulating the given problem, we have

\[
\text{Max } Z = x + 2(y - y')
\]

Subject to \(2x \leq 3y - y' \leq 4\)
\(3x \leq 4y - y' \leq 5\)
\(x, y, y' \geq 0\)

Since the problem is not in the canonical form we rearrange the constraints.

\[
\text{Max } Z = x + 2y' - 2y''
\]

Subject to \(2x \leq 3y \leq 3y' \leq 4\)
\(3x \leq 4y \leq 4y' \leq 5\)
\(x, y, y', y'' \geq 0\)

**Dual:** Since there are three variables and three constraints, in dual we have three variables, namely \(w_1, w_2', w_2''\).

\[
\text{Min } Z = 4w_1' + 5w_2' + 5w_2''
\]

Subject to \(2w_1 + 3w_2' + 3w_2'' \leq 1\)
\(3w_1' + 4w_2' + 4w_2'' \leq 2\)
\(3w_1' + 4w_2' + 4w_2'' \leq 2\)
\(w_1', w_2', w_2'' \geq 0\)

Let \(w_2 = w_2' - w_2''\), so that the dual variable \(w_2\) is unrestricted in sign. Finally the dual is

\[
\text{Min } Z^1 = 4w_1' + 5(w_2 - 5w_2'')
\]

Subject to \(2w_1' + 3(w_2 - w_2'') \leq 1\)
\(3w_1' + 4(w_2 - w_2'') \leq 2\)
\(3w_1' + 4(w_2 - w_2'') \leq 2\)

i.e., \(\text{Min } Z^1 = -4w_1' + 5w_2''\)

Subject to \(2w_1' + 3w_2' \leq 1\)
\(3w_1' + 4w_2' \leq 2\)
\(3w_1' + 4w_2' \leq 2\)
\( w_1 \geq 0 \) and \( w_2 \) is unrestricted.

i.e., \( \text{Min } Z^1 = -4w_1 + 5w_2 \)

Subject to \[
\begin{align*}
w_1 & \geq 0 \\
3w_1 & = 2 \\
3w_1 & = 2
\end{align*}
\]

i.e., \( \text{Min } Z^1 = -4w_1 + 5w_2 \)

Subject to \[
\begin{align*}
-2w_1 + 3w_2 & \geq 1 \\
-3w_1 + 4w_2 & = 2, \quad w_1 \geq 0 \text{ and } w_2 \text{ is unrestricted.}
\end{align*}
\]

**Example 2.29:** Write the dual of the following primal LPP.

Min \( Z = 4x_1 + 5x_2 - 3x_3 \)

Subject to \[
\begin{align*}
x_1 & + x_2 \geq 0 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

\( x_1 + x_2 \geq 0 \) and \( x_3 \) is unrestricted.

**Solution:** Since the variable \( x_3 \) is unrestricted, \( x_1, x_2, x_3 \). Also, bring the problem into the canonical form by rearranging the constraints.

Min \( Z = 4x_1 + 5x_2 - 3(x_3, x_3) \)

Subject to \[
\begin{align*}
x_1 & + (x_3, x_3) \geq 22 \\
x_1, x_2, x_3, x_3 & \geq 22 \\
3x_1 & + 5x_2 = 2(x_3, x_3) \\
x_1, 7x_2, 4(x_3, x_3) & \geq 120 \\
x_1, x_2, x_3, x_3 & = 0
\end{align*}
\]

Min \( Z = 4x_1 + 5x_2 - 3x_3 \)

Subject to \[
\begin{align*}
x_1 & + x_2 \geq 0 \\
x_1, x_2, x_3, x_3 & \geq 0
\end{align*}
\]

\( x_1, x_2, x_3 \)

**Dual:** Since there are four constraints in the primal problem, in dual there are four variables, namely \( w_1, w_2, w_3, w_4 \) so that the dual is given by

Max \( Z \) \( 22(w_1 + w_1) \) \( 65w_2 \) \( 120w_3 \)
Let \( w_i \), \( w_i \), i.e., the variable \( w_i \) is unrestricted.

i.e. \( \text{Max } Z = 22(w_1 w_i) + 65w_2 + 120w_3 \)

Subject to
\[
\begin{align*}
& w_1 w_i 3w_2 w_3 4 \\
& w_1 w_i 5w_2 7w_3 5 \\
& (w_i w_i) 2w_2 4w_3 3 \\
& (w_i w_i) 2w_2 4w_3 3 \\
\end{align*}
\]

i.e., \( \text{Max } Z = 22w_i + 65w_2 + 120w_3 \)

Finally we have

\( \text{Min } Z = 22w_i + 65w_2 + 120w_3 \)

Subject to
\[
\begin{align*}
& w_1 3w_2 w_3 4 \\
& w_1 5w_2 7w_3 5 \\
& w_i 2w_2 4w_3 3 \\
& w_i 2w_2 4w_3 3 \\
\end{align*}
\]

\( w_i, w_1, w_2, w_3 \ge 0 \) and \( w_i \) is unrestricted.

### 2.6.3 Important Results in Duality

1. The dual of the dual is primal.
2. If one is a maximization problem then the other is a minimization one.
3. The necessary and sufficient condition for any LPP and its dual to have an optimal solution is that both must have feasible solution.
4. Fundamental duality theorem states if either the primal or dual problem has a finite optimal solution, then the other problem also has a finite optimal solution and also the optimal values of the objective function in both the problems are the same i.e. \( \text{Max } Z = \text{Min } Z \). The solution of the other problem can be read from the \( Z_j - C_j \) row below the columns of
5. Existence Theorem states that, if either problem has an unbounded solution then the other problem has no feasible solution.

6. Complementary slackness theorem: According to which
   (i) If a primal variable is positive, then the corresponding dual constraint is an equation at the optimum and vice versa.
   (ii) If a primal constraint is a strict inequality then the corresponding dual variable is zero at the optimum and vice versa.

**Example 2.30:** Find the maximum of \( Z = 6x + 8y \)

Subject to
\[
\begin{align*}
5x & \leq 20 \\
x & \leq 10 \\
x, y & \geq 0
\end{align*}
\]

by solving its dual problem.

**Solution:** The dual of the given primal problem is given below. As there are two constraints in the primal, we have two dual variables, namely \( w_1 \) and \( w_2 \).

Min \( Z' = 20w_1 + 10w_2 \)

Subject to
\[
\begin{align*}
5w_1 & \leq 6 \\
2w_1 & \leq 8 \\
w_1, w_2 & \geq 0
\end{align*}
\]

We solve the dual problem using the big M method. Since this method involves artificial variables, the problem is reformulated and we have

\[
\begin{align*}
\text{Max } Z & = 20w_1 + 10w_2 \quad 0S_1 \quad 0S_2 \quad MA_1 \quad MA_2 \\
\text{Subject to } & 5w_1 \quad w_2 \quad S_1 \quad A_1 \quad 6 \\
& 2w_1 \quad 2w_2 \quad S_2 \quad A_2 \quad 8 \\
& w_1, w_2, S_1, S_2, A_1, A_2 \geq 0
\end{align*}
\]

\[
\begin{array}{c|cccccccc}
C_j & 1 & 1 & 3 & 0 & 0 & 0 & 0 \\
\hline
C_B & B & X_B & w_1 & w_2 & S_1 & S_2 & A_1 & A_2 & \text{Max } X_B \\
\hline
\leftarrow M & A_1 & 6 & 3 & 1 & -1 & 0 & 1 & 0 & 6/5 = 1.02 \\
\rightarrow M & A_2 & 8 & 2 & 0 & -1 & 0 & 1 & 8/2 = 4 \\
\hline
Z_j & -14M & -7M & -3M & M & M & -M & -M \\
Z_j - C_j & -7M + 20 & -3M - 10 & M & M & 0 & 0 \\
\uparrow & & & & & & & & \\
-20 & w_1 & 6/5 & 1 & 1/5 & -1/5 & 0 & 0 & 6/5 \times 5/1 = 6
\end{array}
\]
Since all \( Z_j - C_j \geq 0 \), the solution is optimum. Therefore, the optimal solution of dual is
\[
w_1 = \frac{1}{2}, \quad w_2 = \frac{7}{2}, \quad \min Z' = -45
\]

The optimum solution of the primal problem is given by the value of \( Z_j - C_j \) in the optimal table corresponding to the column surplus variables \( S_1 \) and \( S_2 \).

\[
\therefore x = \frac{5}{2}, \quad y = \frac{15}{4}
\]

Max \( Z = 6 \times \frac{5}{2} + 8 \times \frac{15}{4} = 45 \)

**Example 2.31:** Apply the principle of duality to solve the LPP.

Max \( Z = 3x_1 + 2x_2 \)

subject to \( x_1 + x_2 \geq 1 \)
\[x_1 + x_2 \leq 10 \]
\[x_2 \leq 3, \quad x_1, x_2 \geq 0 \]

Solution: First we convert the given (primal) problem into its dual. As there are four constraints in the primal problem we have four variables \( w_1, w_2, w_3, w_4 \) in its dual. We convert the given problem into its canonical form by rearranging some of the constraints.

Max \( Z = 3x_2 + 2x_1 \)

Subject to \(-x_1 - x_2 \leq -1 \)
\[x_1 + x_2 \leq 7 \]
\[x_1 + 2x_2 \leq 10 \]
\[ 0x_1 + x_2 \leq 3 \]
\[ x_1, x_2 \geq 0 \]

**Dual**

\[ \text{Min } Z' = -w_1 + 7w_2 + 10w_3 + 3w_4 \]

Subject to
\[ -w_1 + w_2 + w_3 + 0w_4 \geq 3 \]
\[ -w_1 + w_2 + 2w_3 + w_4 \geq 2 \]
\[ w_1, w_2, w_3, w_4 \geq 0 \]

We apply Big M method to get the solution of the dual problem, as it involves artificial variables.

\[ \text{Max } Z' = w_1 - 7w_2 - 10w_3 - 3w_4 + 0S_1 + 0S_2 - MA_1 - MA_2 \]

Subject to
\[ -w_1 + w_2 + w_3 + 0w_4 - S_1 + A_1 = 3 \]
\[ -w_1 + w_2 + 2w_3 + w_4 - S_2 + A_2 = 2 \]
\[ w_1, w_2, w_3, w_4, S_1, S_2, A_1, A_2 \geq 0 \]

Since all \( Z_j - C_j \geq 0 \) (refer Table 2.1) the solution is optimum. The optimal solution of the dual problem is
\[ w_1 = w_3 = w_4 = 0, w_2 = 3, \text{ Min } Z' = -21. \]

Also from the optimum simplex table of the dual problem, the optimal solution of the primal problem is given by the value \( Z_j - C_j \) corresponding to the column of surplus variables \( S_1 \) and \( S_2 \)

\[ \therefore \quad x_1 = 7, x_2 = 0. \text{ Max } Z = 21. \]

**Example 2.32:** Write down the dual of the following LPP and solve it. Hence or otherwise write down the solution of the primal.

\[ \text{Max } Z = 4x_1 + 2x_2 \]

Subject to
\[ x_1 + x_2 \geq 3 \]
\[ x_1 - x_2 \geq 2 \]
\[ x_1, x_2 \geq 0 \]

**Solution** The dual of the given (primal) problem is as follows. First we convert the given problem into its canonical form by rearranging the constraints.

\[ \text{Max } Z' = 4x_1 + 2x_2 \]

Subject to
\[ -x_1 - x_2 \leq -3 \]
\[ -x_1 + x_2 \leq -2 \]
Table 2.1
\[ x_1, x_2 \geq 0 \]

**Dual**

\[
\text{Min } Z' = -3w_1 - 2w_2
\]

Subject to

- \[ w_1 - w_2 \geq 4 \]
- \[ w_1 + w_2 \geq 2 \]
- \[ w_1, w_2 \geq 0 \]

Introducing the surplus variables \( S_1, S_2 > 0 \) and artificial variables \( A_1, A_2 \geq 0 \) the problem becomes

\[
\text{Max } Z = 3w_1 + 2w_2 + 0S_1 + 0S_2 - MA_1 - MA_2
\]

Subject to

- \[ -w_1 - w_2 + S_1 + A_1 = 4 \]
- \[ w_1 + w_2 - S_2 + A_2 = 2 \]
- \[ w_1, w_2, S_1, S_2, A_1, A_2 \geq 0 \]

<table>
<thead>
<tr>
<th>( C_j )</th>
<th>( B )</th>
<th>( x_1 )</th>
<th>( w_1 )</th>
<th>( w_2 )</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( \text{Min } x_1/x_2 )</th>
</tr>
</thead>
<tbody>
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<td>( -M )</td>
<td>( A_1 )</td>
<td>4</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( 2/1=2 )</td>
<td></td>
</tr>
</tbody>
</table>

\[
\therefore Z_j - C_j \geq 0 \text{ and a artificial variable } A_1 \text{ is in the basis at positive level. Thus, the dual problem does not possess any optimum basic feasible solution. Consequently, there exists no finite optimum solution to the given LP problem or the solution of the given LPP is unbounded.}

**Example 2.33**: Prove using duality theory that the following LPP is feasible but has no optimal solution.

\[
\text{Min } Z = x_1 - x_2 + x_3
\]

Subject to

- \[ x_1 - x_2 + 2x_3 \geq 4 \]
- \[ x_1, x_2, x_3 \geq 0 \]

**Solution**: Given primal LPP is

\[
\text{Min } Z = x_1 - x_2 + x_3
\]

Subject to

- \[ x_1 + 0x_3 - x_2 \geq 4 \]
- \[ x_1 - x_2 + 2x_3 \geq 3 \]
- \[ x_1, x_2, x_3 \geq 0 \]

**Check Your Progress**

4. What are unbounded solutions?
5. Define slack variables.
6. State the complementary slackness theorem.
**Dual** Since there are two constraints, there are two variables $w_1, w_2$ in the dual, given by

\[
\begin{align*}
\text{Max} & \quad Z' = 4w_1 + 3w_2 \\
\text{Subject to} & \quad w_1 + w_2 \leq 1 \\
& \quad 0w_1 - w_2 \leq -1 \\
& \quad -w_1 + 2w_2 \leq 1 \\
& \quad w_1, w_2 \geq 1
\end{align*}
\]

To solve the dual problem

\[
\begin{align*}
\text{Max} & \quad Z' = 4w_1 + 3w_2 \\
\text{Subject to} & \quad w_1 + w_2 + S_1 = 1 \\
& \quad 0w_1 + w_2 - S_2 + A_1 = 1 \\
& \quad -w_1 + 2w_2 + S_3 = 1
\end{align*}
\]

where $S_1, S_3$ are the slack variables, $S_2$ the surplus variable and $A_1$ the artificial variable.

Since all $Z_j - C_j \geq 0$ and an artificial variable appears in the basis at positive level, the dual problem has no optimal basic feasible solution.

\[\therefore\] There exists no finite optimum solution to the given primal LPP (Unbounded solution).
2.7 SUMMARY

This unit has introduced you to linear programming, which is a decision-making technique under given constraints on the assumption that the relationships among the variables representing different phenomena happen to be linear. You have learned that any solution to a LPP which satisfies the non-negativity restrictions of the LPP is called its feasible solution. You have also learned about the simplex method which provides an algorithm that involves moving from one vertex to another. The unit has explained the duality theorem which states that for every maximization problem in LP, there is a unique and similar problem of minimization involving the same data which describes the original problem.

2.8 KEY TERMS

- **Linear programming**: A mathematical technique which involves the allocation of limited resources in an optimal manner on the basis of a given criterion of optimality.
- **Solution**: A set of values \( X_1, X_2 \ldots X_n \) which satisfies the constraints of the LPP is called its solution.
- **Feasible solution**: Any solution to a LPP which satisfies the non-negativity restrictions of the LPP is called its feasible solution.
- **Optimal solution**: Any feasible solution which optimizes (Minimizes or Maximizes) the objective function of the LPP is called its optimum solution.
- **Non-degenerate**: A non-degenerate basic feasible solution is the basic feasible solution which has exactly \( m \) positive \( X_i \) (\( i = 1, 2 \ldots m \)), i.e., none of the basic variables are zero.
- **Degenerate**: A basic feasible solution is said to degenerate if one or more basic variables are zero.
- **Unbounded solution**: If the value of the objective function \( Z \) can be increased or decreased indefinitely, such solutions are called unbounded solutions.
- **Basic solution**: Given a system of \( m \) linear equations with \( n \) variables (\( m < n \)) any solution which is obtained by solving for \( m \) variables keeping the remaining \( n-m \) variables zero is called a basic solution.
- **Fundamental duality theorem**: It states if either the primal or dual problem has a finite optimal solution, then the other problem also has a finite optimal solution and also, the optimal values of the objective function in both the problems are the same, i.e., \( \text{Max } Z = \text{Min } Z \).

2.9 ANSWERS TO ‘CHECK YOUR PROGRESS’

1. Linear programming is a mathematical technique which involves the allocation of limited resources in an optimal manner on the basis of a given criterion of optimality.
2. The general formulation of the LPP can be stated as follows: Maximize or Minimize
\[ Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n \]

3. The linear programming problem can be expressed in the matrix form as follows:
Maximize or Minimize \( Z = CX \)
Subject to \[ \begin{align*}
& \leq b \\
& = b \\
& \geq b \\
\end{align*} \]
\( X \geq 0 \)

Where \( x_1 \), \( x_2 \), \ldots, \( x_n \)
\[ b = \begin{pmatrix} h_1 \\ b_2 \\ \vdots \\ h_m \end{pmatrix} \]
\( C = (C_1, C_2, \ldots, C_n) \)
and \( A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix} \)

4. If the value of the objective function \( Z \) can be increased or decreased indefinitely, such solutions are called \textit{unbounded solutions}.

5. Slack variables are also defined as the non-negative variables which are added in the LHS of the constraint to convert the inequality ‘\( \leq \)’ into an equation.

6. According to the complementary slackness theorem,
(i) If a primal variable is positive, then the corresponding dual constraint is an equation at the optimum and vice versa.
(ii) If a primal constraint is a strict inequality then the corresponding dual variable is zero at the optimum and vice versa.

2.10 QUESTIONS AND EXERCISES

Short-Answer Questions
1. Briefly describe the graphic and simplex methods of solving a linear programming problem.
2. Write short notes on:
   (a) Limitations of linear programming
(b) Dual of a linear programming model
(c) Goal programming

Long-Answer Questions

1. What is linear programming? What types of problems can linear programming help in solving? What characteristics must a problem have if linear programming is to be used?

2. Why is the simplex method considered superior to the graphic method? Explain.

3. (a) Maximize: \( P = 1.4X_1 + X_2 \)
   Subject to:
   \( X_1 \leq 3 \)
   \( 2X_1 + X_2 \leq 8 \)
   \( 3X_1 + 4X_2 \leq 24 \)
   and
   \( X_1 \geq 0, X_2 \geq 0 \)

4. Solve the following problems by the simplex method:
   (a) Minimize: \( 3X_1 + 2X_2 \)
   Subject to:
   \( 2X_1 + 4X_2 \geq 10 \)
   \( 4X_1 + 2X_2 \geq 10 \)
   \( X_2 \geq 4 \)
   and
   \( X_1, X_2 \geq 0 \)
   (b) Minimize: \( Z = 8X_1 + 3X_2 + 2X_3 \)
   Subject to:
   \( 4X_1 + 2X_2 + X_3 \leq 8 \)
   \( 3X_1 + 2X_1 \leq 10 \)
   \( X_1 + X_2 + X_3 = 4 \)
   and
   \( X_1, X_2, X_3 \geq 0 \)

2.11 FURTHER READING


UNIT 3 TRANSPORTATION PROBLEMS

Structure

3.0 Introduction
3.1 Unit Objectives
3.2 Transportation Problems
3.3 Test for Optimality
3.4 Degeneracy in Transportation Problems
3.5 Unbalanced Transportation
3.6 Assignment Problems
3.7 Travelling Salesman Problem
3.8 Summary
3.9 Key Terms
3.10 Answers to ‘Check Your Progress’
3.11 Questions and Exercises
3.12 Further Reading

3.0 INTRODUCTION

Transportation problems are one of the subclasses of LPPs in which the objective is to transport various quantities of a single homogenous commodity that are initially stored at various origins, to different destinations in such a way that the transportation cost is minimum. In this unit, you will also learn about the test for optimality and degeneracy in transportation, as well as the assignment problem, the objective of which is to assign a number of origins to an equal number of destinations at minimum cost or maximum profit.

3.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Explain and solve transportation problems
- Perform the optimality test using the MODI method
- Resolve degeneracy in transportation problems
- Describe what is unbalanced transportation
- Solve assignment problems using the Hungarian Method

3.2 TRANSPORTATION PROBLEMS

The transportation problem is one of the subclasses of LPPs in which the objective is to transport various quantities of a single homogeneous commodity that are initially stored at various origins, to different destinations in such a way that the transportation...
cost is minimum. To achieve this objective we must know the amount and location of available supplies and the quantities demanded. In addition, we must know the costs that result from transporting one unit of commodity from various origins to various destinations.

### Mathematical Formulation

Consider a transportation problem with \( m \) origins (rows) and \( n \) destinations (columns). Let \( C_{ij} \) be the cost of transporting one unit of the product from the \( i^{th} \) origin to \( j^{th} \) destination, \( a_i \) the quantity of commodity available at origin \( i \), \( b_j \) the quantity of commodity needed at destination \( j \), \( x_{ij} \) is the quantity transported from \( i^{th} \) origin to \( j^{th} \) destination. This following transportation problem can be stated in the tabular form.

<table>
<thead>
<tr>
<th>Origins</th>
<th>Destinations</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>\ldots</th>
<th>( n )</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_{11} )</td>
<td>( C_{11} )</td>
<td>( x_{12} )</td>
<td>( C_{12} )</td>
<td>( x_{13} )</td>
<td>( C_{13} )</td>
<td>( x_{1n} )</td>
</tr>
<tr>
<td>2</td>
<td>( x_{21} )</td>
<td>( C_{21} )</td>
<td>( x_{22} )</td>
<td>( C_{22} )</td>
<td>( x_{23} )</td>
<td>( C_{23} )</td>
<td>( x_{2n} )</td>
</tr>
<tr>
<td>3</td>
<td>( x_{31} )</td>
<td>( C_{31} )</td>
<td>( x_{32} )</td>
<td>( C_{32} )</td>
<td>( x_{33} )</td>
<td>( C_{33} )</td>
<td>( x_{3n} )</td>
</tr>
<tr>
<td>( m )</td>
<td>( x_{m1} )</td>
<td>( C_{m1} )</td>
<td>( x_{m2} )</td>
<td>( C_{m2} )</td>
<td>( x_{m3} )</td>
<td>( C_{m3} )</td>
<td>( x_{mn} )</td>
</tr>
</tbody>
</table>

| Demand | \( b_1 \) | \( b_2 \) | \( b_3 \) | \( \ldots \) | \( b_n \) | \[ \sum_{j=1}^{m} a_i = \sum_{j=1}^{n} b_j \]

The linear programming model representing the transportation problem is given by

Minimize \( Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} \)

subject to the constraints

\[ \sum_{j=1}^{n} x_{ij} = a_i \quad i = 1,2,\ldots,n \]

(Row Sum)

\[ \sum_{i=1}^{m} x_{ij} = b_j \quad j = 1,2,\ldots,n \]

(Column Sum)

\[ x_{ij} \geq 0 \quad \text{for all } i \text{ and } j \]

The given transportation problem is said to be balanced if

\[ \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \]

ie., if the total supply is equal to the total demand.
Definitions

**Feasible Solution** Any set of non-negative allocations \(x_{ij} > 0\) which satisfies the row and column sum (rim requirement) is called a feasible solution.

**Basic Feasible Solution** A feasible solution is called a basic feasible solution if the number of non-negative allocations is equal to \(m \times n[1]\), where \(m\) is the number of rows and \(n\) the number of columns in a transportation table.

**Non-degenerate Basic Feasible Solution** Any feasible solution to a transportation problem containing \(m\) origins and \(n\) destinations is said to be non-degenerate, if it contains \(m \times n[1]\) occupied cells and each allocation is in independent positions.

The allocations are said to be in independent positions, if it is impossible to form a closed path.

Closed path means by allowing horizontal and vertical lines and all the corner cells are occupied.

The allocations in the following tables are not in independent positions.

![Table 1](image1)

![Table 2](image2)

The allocations in the following tables are in independent positions.

![Table 3](image3)

![Table 4](image4)

**Degenerate Basic Feasible Solution** If a basic feasible solution contains less than \(m \times n[1]\) non-negative allocations, it is said to be degenerate.

**Optimal Solution**

Optimal solution is a feasible solution (not necessarily basic) which minimizes the total cost. The solution of a transportation problem can be obtained in two stages, namely initial solution and optimum solution.
Initial solution can be obtained by using any one of the three methods, viz,

(i) North west corner rule (NWCR)

(ii) Least cost method or matrix minima method

(iii) Vogel’s approximation method (VAM)

VAM is preferred over the other two methods, since the initial basic feasible solution obtained by this method is either optimal or very close to the optimal solution.

The cells in the transportation table can be classified as occupied cells and unoccupied cells. The allocated cells in the transportation table are called occupied cells and the empty cells in a transportation table are called unoccupied cells.

The improved solution of the initial basic feasible solution is called optimal solution which is the second stage of solution, that can be obtained by MODI (modified distribution method).

**North West Corner Rule**

**Step 1** Starting with the cell at the upper left corner (North west) of the transportation matrix we allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied i.e., \( X_{11} = \min(a_1, b_1) \).

**Step 2** If \( b_1 > a_1 \), we move down vertically to the second row and make the second allocation of magnitude \( x_{22} = \min(a_2, b_1 - x_{11}) \) in the cell (2, 1).

If \( b_1 < a_1 \), move right horizontally to the second column and make the second allocation of magnitude \( X_{12} = \min(a_1, X_{11}, b_1) \) in the cell (1, 2).

If \( b_1 = a_1 \) there is a tie for the second allocation. We make the second allocations of magnitude \( x_{12} = \min(a_1 - a_1, b_1) = 0 \) in cell (1, 2).

or \( x_{21} = \min(a_2 - a_1 - b_1) = 0 \) in cell (2, 1).

**Step 3** Repeat steps 1 and 2 moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

**Example 3.1** Obtain the initial basic feasible solution of a transportation problem whose cost and rim requirement table is the following.

<table>
<thead>
<tr>
<th>Origin/Destination</th>
<th>( D_1 )</th>
<th>( D_2 )</th>
<th>( D_3 )</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_1 )</td>
<td>2</td>
<td>7</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>( O_2 )</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>( O_3 )</td>
<td>5</td>
<td>4</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>( O_4 )</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>Demand</td>
<td>7</td>
<td>9</td>
<td>18</td>
<td>34</td>
</tr>
</tbody>
</table>

**Solution** Since \( \Sigma a_i = 34 = \Sigma b_j \) there exists a feasible solution to the transportation problem. We obtain initial feasible solution as follows.

The first allocation is made in the cell (1, 1) the magnitude being \( x_{11} = \min(5, 7) = 5 \). The second allocation is made in the cell (2, 1) and the magnitude of the allocation is given by \( x_{21} = \min(8, 7 - 5) = 2 \).
The third allocation is made in the cell (2, 2) the magnitude \( x_{22} = \min (8 - 2, 9) = 6. \)

The magnitude of fourth allocation is made in the cell (3, 2) given by \( \min (7, 9 - 6) = 3. \)

The fifth allocation is made in the cell (3, 3) with magnitude \( x_{33} = \min (7 - 3, 14) = 4. \)

The final allocation is made in the cell (4, 3) with magnitude \( x_{43} = \min (14, 18 - 4) = 14. \)

Hence, we get the initial basic feasible solution to the given T.P. and is given by
\[
X_{11} = 5; X_{21} = 2; X_{22} = 6; X_{32} = 3; X_{33} = 4; X_{43} = 14.
\]

Total cost = \( 2 \times 5 - 3 \times 2 - 3 \times 6 - 3 \times 4 - 4 \times 7 - 2 \times 14 \)
\[= 10 \times 6 - 18 \times 12 - 28 - 28 = \text{Rs 102}. \]

**Example 3.2** Determine an initial basic feasible solution to the following transportation problem using N.W.C.R

<table>
<thead>
<tr>
<th></th>
<th>( D_1 )</th>
<th>( D_2 )</th>
<th>( D_3 )</th>
<th>( D_4 )</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_1 )</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>( O_2 )</td>
<td>8</td>
<td>9</td>
<td>2</td>
<td>7</td>
<td>16</td>
</tr>
<tr>
<td>( O_3 )</td>
<td>4</td>
<td>3</td>
<td>6</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Required</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>4</td>
<td>35</td>
</tr>
</tbody>
</table>

**Solution** The problem is a balanced TP as the total supply is equal to the total demand. Using the steps involved we find the initial basic feasible solution as given in the following table

<table>
<thead>
<tr>
<th></th>
<th>( D_1 )</th>
<th>( D_2 )</th>
<th>( D_3 )</th>
<th>( D_4 )</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_1 )</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>( O_2 )</td>
<td>8</td>
<td>9</td>
<td>2</td>
<td>7</td>
<td>16</td>
</tr>
<tr>
<td>( O_3 )</td>
<td>4</td>
<td>3</td>
<td>6</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Demand</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>4</td>
<td>35</td>
</tr>
</tbody>
</table>

The solution is given by
\[
X_{11} = 6; X_{12} = 8; X_{23} = 14; X_{33} = 1; X_{34} = 4.
\]

Total Cost \( 6 \times 6 + 4 \times 8 + 2 \times 9 + 2 \times 14 + 6 \times 1 + 2 \times 4 = \text{Rs 128} \)
Self-Instructional Material

NOTES

Least Cost or Matrix Minima Method

**Step 1** Determine the smallest cost in the cost matrix of the transportation table. Let it be \( C_{ij} \). Allocate \( x_{ij} = \min (a_i, b_j) \) in the cell \((i, j)\).

**Step 2** If \( x_{ij} = a_i \) cross off the \( i^{th} \) row of the transportation table and decrease \( b_j \) by \( a_i \). Then go to step 3.

If \( x_{ij} = b_j \) cross off the \( j^{th} \) column of the transportation table and decrease \( a_i \) by \( b_j \). Go to step 3.

If \( x_{ij} = a_i = b_j \) cross off either the \( i^{th} \) row or the \( j^{th} \) column but not both.

**Step 3** Repeat steps 1 and 2 for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

**Example 3.3** Obtain an initial feasible solution to the following TP using Matrix Minima Method.

<table>
<thead>
<tr>
<th></th>
<th>( D_1 )</th>
<th>( D_2 )</th>
<th>( D_3 )</th>
<th>( D_4 )</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_1 )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>( O_2 )</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>( O_3 )</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>Demand</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>6</td>
<td>24</td>
</tr>
</tbody>
</table>

**Solution** Since \( \Sigma a_i = \Sigma b_j = 24 \), there exists a feasible solution to the TP using the steps in the least cost method, the first allocation is made in the cell \((3, 1)\) the magnitude being \( x_{31} = 4 \). This satisfies the demand at the destination \( D_1 \) and we delete this column from the table as it is exhausted.

<table>
<thead>
<tr>
<th></th>
<th>( D_1 )</th>
<th>( D_2 )</th>
<th>( D_3 )</th>
<th>( D_4 )</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_1 )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>( O_2 )</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>( O_3 )</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>Demand</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>6</td>
<td>24</td>
</tr>
</tbody>
</table>

The second allocation is made in the cell \((2, 4)\) with magnitude \( x_{24} = \min (6, 8) = 6 \). Since it satisfies the demand at the destination \( D_4 \), it is deleted from the table. From the reduced table, the third allocation is made in the cell \((3, 3)\) with magnitude \( x_{33} = \min (8, 6) = 6 \). The next allocation is made in the cell \((2, 3)\) with magnitude \( x_{23} = \min (2, 2) = 2 \). Finally, the allocation is made in the cell \((1, 2)\) with magnitude \( x_{12} = \min (6, 6)/6 \). Now all the rim requirements have been satisfied and hence, the initial feasible solution is obtained.

The solution is given by

\[ x_{12} = 6, x_{23} = 2, x_{24} = 6, x_{31} = 4, x_{33} = 6 \]

Since the total number of occupied cells = \( 5 < m + n - 1 \).

We get a degenerate solution.

Total cost = \( 6 \times 2 + 2 \times 2 + 6 \times 0 + 4 \times 0 + 6 \times 2 \)
\[ = 12 + 4 + 12 = Rs \, 28. \]
**Example 3.4** Determine an initial basic feasible solution for the following TP, using the least cost method.

<table>
<thead>
<tr>
<th></th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>$O_2$</td>
<td>8</td>
<td>9</td>
<td>2</td>
<td>2</td>
<td>16</td>
</tr>
<tr>
<td>$O_3$</td>
<td>4</td>
<td>3</td>
<td>6</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Demand</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>4</td>
<td>35</td>
</tr>
</tbody>
</table>

**Solution** Since $\sum a_i = \sum b_j$, there exists a basic feasible solution. Using the steps in least cost method we make the first allocation to the cell $(1, 3)$ with magnitude $X_{13} = \min(14, 15) = 14$. (as it is the cell having the least cost).

This allocation exhausts the first row supply. Hence, the first row is deleted. From the reduced table, the next allocation is made in the next least cost cell $(2, 3)$ which is chosen arbitrarily with magnitude $X_{23} = \min(1, 16) = 1$. This exhausts the 3rd column destination.

From the reduced table the next least cost cell is $(3, 4)$ for which allocation is made with magnitude $\min(4, 5) = 4$. This exhausts the destination $D_4$ requirement. Delete this fourth column from the table. The next allocation is made in the cell $(3, 2)$ with magnitude $X_{32} = \min(1, 10) = 1$. This exhausts the 3rd origin capacity. Hence, the 3rd row is exhausted. From the reduced table the next allocation is given to the cell $(2, 1)$ with magnitude $X_{21} = \min(6, 15) = 6$. This exhausts the first column requirement. Hence, it is deleted from the table.

Finally, the allocation is made to the cell $(2, 2)$ with magnitude $X_{22} = \min(9, 9) = 9$ which satisfies the rim requirement. These allocation are shown in the transportation table as follows.
The following table gives the initial basic feasible solution.

<table>
<thead>
<tr>
<th>Origin</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>$O_2$</td>
<td>8</td>
<td>9</td>
<td>2</td>
<td>7</td>
<td>16</td>
</tr>
<tr>
<td>$O_3$</td>
<td>4</td>
<td>3</td>
<td>6</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

The solution is given by

$X_{13} = 14; X_{21} = 6; X_{22} = 9; X_{23} = 1; X_{32} = 1; X_{34} = 4$

Transportation cost

$= 14 \times 1 \times 6 + 9 \times 9 + 1 \times 2 + 3 \times 1 + 4 \times 2$

$= Rs 156.$

**Vogel’s Approximation Method (VAM)**

The steps involved in this method for finding the initial solution are as follows.

**Step 1** Find the penalty cost, namely the difference between the smallest and next smallest costs in each row and column.

**Step 2** Among the penalties as found in Step(1) choose the maximum penalty. If this maximum penalty is more than one (i.e., if there is a tie) choose any one arbitrarily.

**Step 3** In the selected row or column as by Step(2) find out the cell having the least cost. Allocate to this cell as much as possible depending on the capacity and requirements.

**Step 4** Delete the row or column which is fully exhausted. Again, compute the column and row penalties for the reduced transportation table and then go to Step (2). Repeat the procedure until all the rim requirements are satisfied.

**Note** If the column is exhausted, then there is a change in row penalty and vice versa.

**Example 3.5** Find the initial basic feasible solution for the following transportation problem by VAM.

<table>
<thead>
<tr>
<th>Origin</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>11</td>
<td>13</td>
<td>17</td>
<td>14</td>
<td>250</td>
</tr>
<tr>
<td>$O_2$</td>
<td>16</td>
<td>18</td>
<td>14</td>
<td>10</td>
<td>300</td>
</tr>
<tr>
<td>$O_3$</td>
<td>21</td>
<td>24</td>
<td>13</td>
<td>10</td>
<td>400</td>
</tr>
</tbody>
</table>

| Demand | 200   | 225   | 275   | 250   | 950    |

**Solution** Since $\Sigma a_i = \Sigma b_i = 950$ the problem is balanced and there exists a feasible solution to the problem.

First, we find the row $\Sigma$ column penalty $P_i$ as the difference between the least and the next least cost. The maximum penalty is 5. Choose the first column arbitrarily.
In this column, choose the cell having the least cost name (1, 1). Allocate to this cell with minimum magnitude (i.e. (250, 200) = 200.) This exhausts the first column. Delete this column. Since a column is deleted, then there is a change in row penalty $P_i$ and column penalty remains the same. Continuing in this manner, we get the remaining allocations as given in the following table below.

Finally, we arrive at the initial basic feasible solution which is shown in the following table.
There are 6 positive independent allocations which equals to $m + n - 1 = 3 + 4 - 1$. This ensures that the solution is a non-degenerate basic feasible solution.

∴ The transportation cost

\[= 11 \times 200 + 13 \times 50 + 18 \times 175 + 10 \times 125 + 13 \times 275 + 10 \times 125 = \text{Rs 12,075}.\]

**Example 3.6** Find the initial solution to the following TP using VAM.

<table>
<thead>
<tr>
<th>Factory</th>
<th>Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>145</td>
</tr>
<tr>
<td>$F_2$</td>
<td>80</td>
</tr>
<tr>
<td>$F_3$</td>
<td>55</td>
</tr>
</tbody>
</table>

Finally, we have the initial basic feasible solution as given in the following table.

<table>
<thead>
<tr>
<th>Destination</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>Supply</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>100</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$F_2$</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>125</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>-</td>
</tr>
<tr>
<td>$F_3$</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>75</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Demand</td>
<td>120</td>
<td>80</td>
<td>75</td>
<td>25</td>
<td>300</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

There are 6 independent non-negative allocations equal to $m|p|[1\ 3]4[\ 6]$. This ensures that the solution is non-degenerate basic feasible.

∴ The transportation cost

\[= 3 \times 45 + 4 \times 30 + 1 \times 25 + 2 \times 80 + 4 \times 45 + 1 + 75\]

\[= 135 + 120 + 25 + 160 + 180 + 75\]

\[= \text{Rs 695}\]
3.3 TEST FOR OPTIMALITY

Once the initial basic feasible solution has been computed, the next step in the problem is to determine whether the solution obtained is optimum or not.

Optimality test can be conducted to any initial basic feasible solution of a TP provided such allocations has exactly \( m + n - 1 \) non-negative allocations, where \( m \) is the number of origins and \( n \) is the number of destinations. Also, these allocations must be in independent positions.

To perform this optimality test, we shall discuss the modified distribution method (MODI). The various steps involved in the MODI method for performing optimality test are as follows.

**MODI Method**

**Step 1** Find the initial basic feasible solution of a TP by using any one of the three methods.

**Step 2** Find out a set of numbers \( u_i \) and \( v_j \) for each row and column satisfying \( u_i + v_j = c_{ij} \) for each occupied cell. To start with, we assign a number ‘0’ to any row of column having maximum number of allocations. If this maximum number of allocations is more than one, choose any one arbitrarily.

**Step 3** For each empty (unoccupied) cell, we find the sum \( u_i \) and \( v_j \) written in the bottom left corner of that cell.

**Step 4** Find out for each empty cell the net evaluation value \( \Delta_{ij} = c_{ij} - (u_i + v_j) \) and which is written at the bottom right corner of that cell. This step gives the optimality conclusion,

(i) If all \( \Delta_{ij} > 0 \) (i.e., all the net evaluation value) the solution is optimum and a unique solution exists.

(ii) If \( \Delta_{ij} \leq 0 \) then the solution is optimum, but an alternate solution exists.

(iii) If at least one \( \Delta_{ij} < 0 \), the solution is not optimum. In this case, we go to the next step, to improve the total transportation cost.

**Step 5** Select the empty cell having the most negative value of \( \Delta_{ij} \). From this cell we draw a closed path by drawing horizontal and vertical lines with the corner cells occupied. Assign sign ] and [ alternately and find the minimum allocation from the cell having negative sign. This allocation should be added to the allocation having positive sign and subtracted from the allocation having negative sign.

**Step 6** The previous step yields a better solution by making one (or more) occupied cell as empty and one empty cell as occupied. For this new set of basic feasible allocations, repeat from the step(2) till an optimum basic feasible solution is obtained.

**Example 3.7** Solve the following transportation problem.

<table>
<thead>
<tr>
<th>Destination</th>
<th>( \text{Supply} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( P )</td>
</tr>
<tr>
<td>( A )</td>
<td>21</td>
</tr>
<tr>
<td>( B )</td>
<td>17</td>
</tr>
<tr>
<td>( C )</td>
<td>32</td>
</tr>
<tr>
<td>( \text{Demand} )</td>
<td>6</td>
</tr>
</tbody>
</table>
NOTES

Solution  We first find the initial basic feasible solution by using VAM. Since $\Sigma a_i = \Sigma b_j$ the given TP is a balanced one. Therefore, there exists a feasible solution.

Finally, we have the initial basic feasible solution as given in the following table.

<table>
<thead>
<tr>
<th>Source</th>
<th>Destination</th>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>21</td>
<td>16</td>
<td>23</td>
<td>13</td>
<td>11</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>17</td>
<td>18</td>
<td>48</td>
<td>19</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td>32</td>
<td>17</td>
<td>18</td>
<td>43</td>
<td>6</td>
</tr>
</tbody>
</table>

From this table, we see that the number of non-negative independent allocations is $6 = m+n+1 = 3+4+1$.

Hence, the solution is non-degenerate basic feasible.

.: The initial transportation cost

$= 11 \times 13 + 3 \times 14 + 4 \times 23 + 6 \times 17 + 17 \times 10 + 18 \times 9 = \text{Rs} 711$

To find the optimal solution  We apply the MODI method in order to determine the optimum solution. We determine a set of numbers $u_i$ and $v_j$ for each row and column, with $u_i + v_j = c_{ij}$ for each occupied cell. To start with we give $u_2 = 0$ as the 2nd row has the maximum number of allocation.

Now, we find the sum $u_i$ and $v_j$ for each empty cell and enter at the bottom left corner of that cell.

Next, we find the net evaluations $\Delta_{ji} = c_{ij} - (u_i + v_j)$ for each unoccupied cell and enter at the bottom right corner of that cell.

Initial Table

<table>
<thead>
<tr>
<th>Source</th>
<th>Destination</th>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
<th>$U_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>21</td>
<td>16</td>
<td>23</td>
<td>13</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>17</td>
<td>18</td>
<td>48</td>
<td>19</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td>32</td>
<td>17</td>
<td>18</td>
<td>43</td>
<td>4</td>
</tr>
<tr>
<td>$V_j$</td>
<td></td>
<td>17</td>
<td>13</td>
<td>14</td>
<td>16</td>
<td>23</td>
</tr>
</tbody>
</table>
Since all $\Delta_{ij} > 0$, the solution is optimal and unique. The optimum solution is given by

\[ x_{14} = 11, x_{21} = 6, x_{23} = 3, x_{24} = 4, x_{32} = 10, x_{33} = 9 \]

The minimum transportation cost

\[ = 11e_{13} + 17e_{6} + 3e_{14} + 4e_{23} + 10e_{17} + 9e_{18} = \text{Rs } 711. \]

**Example 3.8** Solve the following transportation problem starting with the initial solution obtained by VAM.

### Solution

Since $\sum a_i = \sum b_j$ the problem is a balanced TP. Therefore, there exists a feasible solution.

Finally, the initial basic feasible solution is given as follows:

Since the number of occupied cells = $6 = m - n + 1$ and are also independent, there exists a non-degenerate basic feasible solution.

The initial transportation cost

\[ = 3 e_{2} + 3 e_{5} + 4 e_{4} + 7 e_{1} + 6 e_{3} + 6 e_{1} = \text{Rs } 68 \]

To find the optimal solution, applying the MODI method, we determine a set of numbers $u_i$ and $v_j$ for each row and column, such that $u_i + v_j = c_{ij}$ for each occupied
cell. Since the 3rd now has maximum number of allocations we give number \( u_3 = 0 \). The remaining numbers can be obtained as follows.

\[
\begin{align*}
\hline
\text{cell} & \quad \text{since the 3rd now has maximum number of allocations we give number} \quad u_3 = 0. \quad \text{The remaining numbers can be obtained as follows.} \\
\text{31} & = u_3 + v_1 = 7 = 0 + v_1 = 7 \Rightarrow v_1 = 7 \\
\text{32} & = u_3 + v_2 = 6 = 0 + v_2 = 6 \Rightarrow v_2 = 6 \\
\text{33} & = u_3 + v_3 = 6 = 0 + v_3 = 6 \Rightarrow v_3 = 6 \\
\text{34} & = u_2 + v_4 = 5 = u_2 + 6 = 5 \Rightarrow u_2 = -1 \\
\text{41} & = u_1 + v_1 = 2 - u_1 + 7 = 2 \Rightarrow u_1 = -5 \\
\hline
\end{align*}
\]

We find the sum \( u_j \) and \( v_j \) for each empty cell and enter at the bottom left corner of the cell. Next, we find the net evaluation \( \Delta_{ij} \) given by \( \Delta_{ij} = C_{ij} - (u_j + v_i) \) for each empty cell and enter at the bottom right corner of the cell.

**Initial table**

<table>
<thead>
<tr>
<th>\text{D}</th>
<th>\text{D}</th>
<th>\text{D}</th>
<th>\text{D}</th>
<th>\text{u}</th>
<th>\text{v}</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{1}</td>
<td>\text{2}</td>
<td>\text{3}</td>
<td>\text{4}</td>
<td>\text{5}</td>
<td></td>
</tr>
<tr>
<td>\text{O}_1</td>
<td>\text{2}</td>
<td>\text{1}</td>
<td>\text{1}</td>
<td>\text{1}</td>
<td>u_i = -1</td>
</tr>
<tr>
<td>\text{O}_2</td>
<td>\text{10}</td>
<td>\text{2}</td>
<td>\text{5}</td>
<td>\text{1}</td>
<td>u_i = -1</td>
</tr>
<tr>
<td>\text{O}_3</td>
<td>\text{4}</td>
<td>\text{3}</td>
<td>\text{3}</td>
<td>\text{5}</td>
<td>u_i = -1</td>
</tr>
<tr>
<td>\text{v}_1 = 7</td>
<td>\text{v}_2 = 6</td>
<td>\text{v}_3 = 6</td>
<td>\text{v}_4 = 5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since all \( \Delta_{ij} > 0 \) the solution is optimum and unique. The solution is given by

\( x_{11} = 3; x_{23} = 3; x_{24} = 4 \)
\( x_{31} = 1; x_{32} = 3; x_{33} = 1 \)

The total transportation cost

\[ = 2 \times 3 + 3 \times 5 + 4 \times 4 + 7 \times 1 + 6 \times 3 + 6 \times 1 = \text{Rs } 68 \]

### 3.4 DEGENERACY IN TRANSPORTATION PROBLEMS

In a TP, if the number of non-negative independent allocations is less than \( m \) \( n \) \( 1 \), where \( m \) is the number of origins (rows) and \( n \) is the number of destinations (columns) there exists a degeneracy. This may occur either at the initial stage or at the subsequent iteration.

To resolve this degeneracy, we adopt the following steps.

1. Among the empty cell, we choose an empty cell having the least cost which is of an independent position. If this cell is more than one, choose any one arbitrarily.
2. To the cell as chosen in step (1) we allocate a small positive quantity \( \varepsilon > 0 \).

The cells containing \( \varepsilon \) are treated like other occupied cells and degeneracy is removed by adding one (more) accordingly. For this modified solution, we adopt the steps involved in MODI method till an optimum solution is obtained.
Example 3.9  Solve the transportation problem for minimization.

\[
\begin{array}{c|ccc|c}
\text{Sources} & \text{1} & \text{2} & \text{3} & \text{Capacity} \\
\hline
\text{1} & 2 & 2 & 3 & 10 \\
\text{2} & 4 & 1 & 2 & 15 \\
\text{3} & 1 & 3 & 1 & 40 \\
\hline
\text{Demand} & 20 & 15 & 30 & 65
\end{array}
\]

**Solution**  Since \( \Sigma a_i \Sigma b \) the problem is a balanced TP. Hence, there exists a feasible solution. We find the initial solution by North West Corner rule as follows.

\[
\begin{array}{c|ccc|c}
& \text{1} & \text{2} & \text{3} & \text{Capacity} \\
\hline
\text{1} & 10 & 2 & 3 & 10 \\
\text{2} & 15 & 1 & 2 & 15 \\
\text{3} & 10 & 3 & 1 & 30 \\
\hline
\text{Demand} & 20 & 15 & 30 & 65
\end{array}
\]

Since the number of occupied cells = 5 = \( m + n - 1 \) and all the allocations are independent, we get an initial basic feasible solution.

The initial transportation cost
\[
= 10 \times 2 + 4 \times 10 + 5 \times 1 + 10 \times 3 + 1 \times 30 = \text{Rs 125}
\]

To find the optimal solution (MODI method)  We use the previous table to apply the MODI method. We find out a set of numbers \( u_i \) and \( v_j \) for which \( u_i + v_j = c_{ij} \), only for occupied cell. To start with, as the maximum number of allocations is 2 in more than one row and column, we choose arbitrarily column 1, and assign a number 0 to this column, i.e., \( v_1 = 0 \). The remaining numbers can be obtained as follows.

\[
\begin{align*}
c_{11} &= u_1 + v_1 = 2 \\
&= u_1 + 0 = 2 \Rightarrow u_1 = 2 \\
c_{21} &= u_2 + v_1 = 4 \\
&= u_2 + 0 = 4 \\
c_{22} &= u_2 + v_2 = 1 \\
&= v_2 = 1 - u_2 = 1 - 4 = -3 \\
c_{32} &= u_3 + v_2 = 3 \\
&= u_3 = 3 - v_2 = 3 - (-3) = 6 \\
c_{33} &= u_3 + v_3 = 1 \\
&= v_3 = 1 - u_3 = 1 - 6 = -5
\end{align*}
\]
We find the sum of $u_i$ and $v_j$ for each empty cell and write at the bottom left corner of that cell. Find the net evaluations $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for each empty cell and enter at the bottom right corner of the cell.

The solution is not optimum as the cell (3, 1) has a negative $\Delta_{ij}$ value. We improve the allocation by making this cell namely (3, 1) as an allocated cell. We draw a closed path from this cell and assign sign + and − alternately. From the cell having negative sign we find the min. allocation given by $\min (10, 10) = 10$. Hence, we get two occupied cells.

(2, 1) (3, 2) becomes empty and the cell (3, 1) is occupied and resulting in a degenerate solution. (Degeneracy in subsequent iteration).

Number of allocated cells $= 4 < m + n - 1 = 5$.

We get degeneracy. To resolve we add the empty cell (1, 2) and allocate $\varepsilon > 0$. This cell namely (1, 2) is added as it satisfies the two steps for resolving the degeneracy. We assign a number 0 to the first row, namely $u_1 = 0$ we get the remaining numbers as follows.

$c_{11} = u_1 + v_1 = 2 \Rightarrow v_1 = 2 - u_1 = 2 - 0 = 2$
$c_{12} = u_1 + v_2 = 2 \Rightarrow v_2 = 2 - u_1 = 2 - 0 = 2$
$c_{31} = u_3 + v_1 = 1 \Rightarrow u_3 = 1 - v_1 = 1 - 2 = -1$
$c_{33} = u_3 + v_3 = 1 \Rightarrow v_3 = 1 - u_3 = 1 - (-1) = 2$
$c_{22} = u_2 + v_2 = 1 \Rightarrow u_2 = 1 - v_2 = 1 - 2 = -1$

Next, we find the sum of $u_i$ and $v_j$ for the empty cell and enter at the bottom left corner of that cell and also the net evaluation $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for each empty cell and enter at the bottom right corner of the cell.

**I Iteration table**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$u_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>1</td>
<td>2</td>
<td>30</td>
</tr>
</tbody>
</table>

**NOTES**

Initial Table
The modified solution is given in the following table. This solution is also optimal and unique as it satisfies the optimality condition that all $\Delta_{ij} > 0$.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>$\varepsilon$</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>15</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Demand</td>
<td>20</td>
<td>15</td>
<td>30</td>
<td>65</td>
</tr>
</tbody>
</table>

$x_{11} = 10; x_{22} = 15; x_{33} = 30; \]
\[
x_{12} = \varepsilon; x_{31} = 10
\]
\[
\vdots \ldots \ldots \ldots \ldots \ldots \ldots x \varepsilon = 15 \times 1 + 10 \times 1 + 30 \times 1
\]
\[
= 75 + 2 \varepsilon = \text{Rs 75}
\]

**Example 3.10** Solve the following transportation problem whose cost matrix is given below.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Origin</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>34</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>7</td>
<td>2</td>
<td>4</td>
<td>19</td>
</tr>
<tr>
<td>Demand</td>
<td>21</td>
<td>25</td>
<td>17</td>
<td>17</td>
<td>80</td>
</tr>
</tbody>
</table>

**Solution** Since $\sum_{i}a_{i} = \sum_{j}b_{j}$ the problem is a balanced transportation problem. Hence, there exists a feasible solution. We find the initial solution by North west corner rule.

We get the total number of allocated cells = 7 = 4 + 4 – 1. As all the allocations are independent, the solution is a non-degenerate solution.
Total transportation cost
\[= 1 \times 21 + 5 \times 13 + 3 \times 12 + 3 \times 1 + 12 \times 2 + 2 \times 2 + 17 \times 4\]
\[= Rs 221\]

**NOTES**

**To find the optimal solution (MODI Method)** We determine a set of numbers \(u_i\) and \(v_j\) for each row and each column with \(u_i + v_j = c_{ij}\) for each occupied cell. To start with we give 0 to the third column as it has the maximum number of allocation.

**Initial Table**

<table>
<thead>
<tr>
<th></th>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
<th>(D)</th>
<th>(u_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(21)</td>
<td>(13)</td>
<td>3</td>
<td>3</td>
<td>3 (= u_i)</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>3</td>
<td>-3</td>
<td>1</td>
<td>2 (= u_i)</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>12</td>
<td>2</td>
<td>3</td>
<td>2 (= u_i)</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>4</td>
<td>12</td>
<td>-2</td>
<td>4 (= u_i)</td>
</tr>
</tbody>
</table>

\(c_{33} = u_3 + v_3 = 2\)
\(u_3 = 2 - v_3 = 2 - 0\)
\(= 2\)

\(c_{43} = u_4 + v_3 = 2\)
\(u_4 = 2 - 0 = 2\)

\(c_{44} = u_4 + v_4 = 4\)
\(v_4 = 4 - 2 = 2\)

\(c_{22} = u_2 + v_2 = 3\)
\(v_2 = 3 - u_2 = 2\)

\(c_{12} = u_1 + v_2 = 5\)
\(u_1 = 5 - v_2 = 3\)

\(c_{11} = u_1 + v_1 = 1\)
\(v_1 = 1 - u_1 = 1 - 3 = -2\)

We find the sum \(u_i\) and \(v_j\) for each empty cell and enter at the bottom left corner and \(\Delta_{ij} = c_{ij} - (u_i + v_j)\) for each empty cell and enter at the bottom right corner of that cell. The solution is not optimum as some of \(\Delta_{ij}\) <0. We choose the most negative \(\Delta_{ij}\) i.e., 2. There is a tie between the cell (1, 4) and (3, 2) but we choose the cell (3, 2) as it has the least cost. From this cell, we draw a closed path and assign ± signs alternately and find the minimum allocation from the cell having − sign.

Thus, we get, Min (12, 12) = 12. Hence, one empty cell (3, 2) becomes occupied and two occupied cells (2, 2) (3, 3) becomes empty resulting in degeneracy (Degeneracy in subsequent iteration). By adding and subtracting this minimum allocation we get the modified allocation as given in the following table. For this modified allocations, we repeat the steps in the MODI method.
The number of allocations = $6 < m + n - 1 = 7$. We add the cell (3,3) as it is the least cost empty cell which is of independent position. Give a small quantity $\varepsilon > 0$. This removes degeneracy. The modified allocation is given in the following table.

The solution is not optimum. The next negative value of $\Delta_{y} = 4$. (the cell (1, 4)).

The minimum allocation is min. (13, $\varepsilon$, 17) = $\varepsilon$. Proceeding in the same manner we have the 2nd iteration table given as follows.

As the solution is not optimum, we improve the solution by using the steps involved in the MODI method. The most negative value of $\Delta_{y} = -2$. Min allocation is min. ($13 - \varepsilon$, 15, $17 - \varepsilon$) = $13 - \varepsilon$. 

\[\begin{array}{cccc}
A & B & C & D \\
1 & \text{(2)} & 1 & - & 5 & 3 & + & 3 \\
2 & 3 & - & 3 & 1 & 2 \\
3 & 0 & 2 & 2 & 3 \\
4 & 2 & 7 & 2 & 4 \\
\text{v_j} & -4 & 0 & 0 & 2 \\
\end{array}\]

\[\begin{array}{cccc}
A & B & C & D \\
1 & \text{(2)} & 1 & - & 5 & 3 & + & 3 \\
2 & 3 & - & 3 & 1 & 2 \\
3 & 0 & 2 & 2 & 3 \\
4 & 2 & 7 & 2 & 4 \\
\text{v_j} & -4 & 0 & 0 & 2 \\
\end{array}\]

\[\begin{array}{cccc}
A & B & C & D \\
1 & \text{(2)} & 1 & - & 5 & 3 & + & 3 \\
2 & 3 & - & 3 & 1 & 2 \\
3 & 0 & 2 & 2 & 3 \\
4 & 2 & 7 & 2 & 4 \\
\text{v_j} & -4 & 0 & 0 & 2 \\
\end{array}\]
NOTES

III Iteration table

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>2+ε</td>
<td>-3</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>7</td>
<td>15</td>
<td>4</td>
</tr>
</tbody>
</table>

Improve the solution by adding and subtracting the new allocation given by min. \((2 + \epsilon), 4\) = \((2\epsilon)\).

IV Iteration table

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>(u_j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>71</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2+ε</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>3+ε</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>7</td>
<td>15+ε</td>
<td>4</td>
<td>-ε</td>
</tr>
</tbody>
</table>

Since all \(\Delta_{p0}\) the solution is optimum (alternate solution exists). The solution is given by

\[X_{11} = 21; X_{14} = 13; X_{22} = 13 - \epsilon = 13; X_{24} = 2 + \epsilon = 2;\]
\[X_{32} = 12 + \epsilon = 12; X_{43} = 17 + \epsilon = 17; X_{44} = 2 - \epsilon = 2\]

Total transportation cost = \(21 \times 1 + 3 \times 3 + 3(13 - \epsilon) + 2(2 + \epsilon) + 2\)

\(12 + \epsilon) + 2(17 + \epsilon) + 4(2 - \epsilon) = 169 - \epsilon = \text{Rs 169}\)

**Example 3.11** A company has three plants \(A, B\) and \(C\), 3 warehouses \(X, Y, Z\). The number of units available at the plants is 60, 70, 80 and the demand at \(X, Y, Z\) are 50, 80, 80 respectively. The unit cost of the transportation is given in the following table.
Find the allocation so that the total transportation cost is minimum.

**Solution**

<table>
<thead>
<tr>
<th>Plants</th>
<th>Warehouses</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>A</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>C</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>Demand</td>
<td>50</td>
<td>80</td>
</tr>
</tbody>
</table>

Since $\Sigma a_i = \Sigma b_j = 210$ the problem is a balanced one. Hence, there exists a feasible solution. Let us find the initial solution by least cost method.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8</td>
<td>7</td>
<td>3</td>
<td>60</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>70</td>
</tr>
<tr>
<td>C</td>
<td>11</td>
<td>3</td>
<td>5</td>
<td>80</td>
</tr>
<tr>
<td>Demand</td>
<td>50</td>
<td>80</td>
<td>80</td>
<td>210</td>
</tr>
</tbody>
</table>

Here, the least cost cell is not unique, i.e., the cells (2, 1) (1, 3) and (3, 2) have the least value 3. So choose the cell arbitrarily. Let us choose the cell (2, 1) and allocate with magnitude min. $(70, 50) = 50$. This exhausts the first column. Now, delete this column. The reduced transportation table is given by

<table>
<thead>
<tr>
<th></th>
<th>Y</th>
<th>Z</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>7</td>
<td>3</td>
<td>60</td>
</tr>
<tr>
<td>B</td>
<td>8</td>
<td>9</td>
<td>20</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>5</td>
<td>80</td>
</tr>
<tr>
<td>Demand</td>
<td>80</td>
<td>80</td>
<td></td>
</tr>
</tbody>
</table>

Continuing in this manner, we finally arrive at the initial solution which is shown in the following table:

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8</td>
<td>7</td>
<td>3</td>
<td>60</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>70</td>
</tr>
<tr>
<td>C</td>
<td>11</td>
<td>3</td>
<td>5</td>
<td>80</td>
</tr>
<tr>
<td>Demand</td>
<td>50</td>
<td>80</td>
<td>80</td>
<td></td>
</tr>
</tbody>
</table>
The number of allocated cells is \(4 < m + n - 1 = 5\), resulting in degeneracy (Initial stage). To remove this degeneracy we add an empty cell (3, 3) whose cost is minimum and is of independent position. Allocate to this cell a small quantity \(\varepsilon > 0\). Hence, we have the initial solution given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8</td>
<td>7</td>
<td>3</td>
<td>60</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>20</td>
</tr>
<tr>
<td>C</td>
<td>11</td>
<td>3</td>
<td>5</td>
<td>80</td>
</tr>
</tbody>
</table>

The total number of occupied cells = 5 = \(m + n + 1\) = 5. These are also of independent position. This solution is non-degenerate.

The solution is given by
\[
X_{13} = 60, \ x_{21} = 50, \ X_{23} = 20, \ X_{32} = 80, \ X_{33} = \varepsilon
\]

The total transportation cost
\[
= 3 \times 60 + 3 \times 50 + 9 \times 20 + 3 \times 80 + 5 \times \varepsilon
\]
\[
= \text{Rs } 750 + 5 \varepsilon = \text{Rs } 750
\]

To find the optimal solution We apply the steps involved in the MODI method to the previous table. We find a set of numbers \(u_i\) and \(v_j\) for which \(u_i + v_j = c_{ij}\) is satisfied for each of the occupied cell. To start with we assign a number 0 to the third column (i.e., \(v_3 = 0\)) as it has the maximum number of allocations. The remaining numbers are obtained as follows.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>(u_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>−3</td>
<td>11</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>50</td>
<td>3</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>C</td>
<td>−1</td>
<td>12</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Since all \(\Delta_{ij} > 0\) we have obtained an optimum solution.

The solution is given by
\[
X_{13} = 60; \ x_{21} = 50; \ X_{23} = 20; \ X_{32} = 80; \ X_{33} = \varepsilon
\]

Total transportation cost
\[
= 30 \times 60 \times 3 \times 50 \times 9 \times 20 \times 3 \times 80 \times 5 \times \varepsilon
\]
\[
= \text{Rs } 750 + 5 \varepsilon = \text{Rs } 750
\]
3.5 UNBALANCED TRANSPORTATION

The given TP is said to be unbalanced if $\sum a_i \neq \sum b_j$, i.e., if the total supply is not equal to the total demand.

There are two possible cases.

Case i: $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$

If the total supply is less than the total demand, a dummy source (row) is included in the cost matrix with zero cost; the excess demand is entered as a rim requirement for this dummy source (origin). Hence, the unbalanced transportation problem can be converted into a balanced TP.

Case ii: $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$

i.e., the total supply is greater than the total demand. In this case, the unbalanced TP can be converted into a balanced TP by adding a dummy destination (column) with zero cost. The excess supply is entered as a rim requirement for the dummy destination.

Example 3.12  Solve the transportation problem when the unit transportation costs, demands and supplies are as given:

<table>
<thead>
<tr>
<th>Origins</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>6</td>
<td>1</td>
<td>9</td>
<td>3</td>
<td>70</td>
</tr>
<tr>
<td>$O_2$</td>
<td>11</td>
<td>5</td>
<td>2</td>
<td>8</td>
<td>55</td>
</tr>
<tr>
<td>$O_3$</td>
<td>10</td>
<td>12</td>
<td>4</td>
<td>?</td>
<td>70</td>
</tr>
<tr>
<td>Demand</td>
<td>85</td>
<td>35</td>
<td>50</td>
<td>45</td>
<td></td>
</tr>
</tbody>
</table>

Solution  Since the total demand $\sum b_j = 215$ is greater than the total supply $\sum a_i = 195$ the problem is an unbalanced TP.

We convert this into a balanced TP by introducing a dummy origin $O_4$ with cost zero and giving supply equal to $215-195=20$ units. Hence, we have the converted problem as follows

<table>
<thead>
<tr>
<th>Origins</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>6</td>
<td>1</td>
<td>9</td>
<td>3</td>
<td>70</td>
</tr>
<tr>
<td>$O_2$</td>
<td>11</td>
<td>5</td>
<td>2</td>
<td>8</td>
<td>55</td>
</tr>
<tr>
<td>$O_3$</td>
<td>10</td>
<td>12</td>
<td>4</td>
<td>7</td>
<td>70</td>
</tr>
<tr>
<td>$O_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>Demand</td>
<td>85</td>
<td>35</td>
<td>50</td>
<td>45</td>
<td>215</td>
</tr>
</tbody>
</table>

As this problem is balanced, there exists a feasible solution to this problem. Using VAM we get the following initial solution.
The initial solution to the problem is given by

There are 7 independent non-negative allocations equal to $m + n - 1$. Hence, the solution is a non-degenerate one. The total transportation cost

$$= 6 \times 65 + 5 \times 1 + 5 \times 30 + 2 \times 25 + 4 \times 25 + 7 \times 45 + 20 \times 0 = Rs \ 1010$$

To find the optimal solution We apply the steps in the MODI method to the previous table.
Since all $\Delta_{ij} \neq 0$ the solution is not optimum, we introduce the cell (3,1) as this cell has the most negative value of $\Delta_{ij}$. We modify the solution by adding and subtracting the min allocation given by min (65, 30, 25). While doing this, the occupied cell (3, 3) becomes empty.

### Iteration Table

<table>
<thead>
<tr>
<th></th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>$u_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>65</td>
<td>1</td>
<td>9</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$O_2$</td>
<td>10</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>$O_3$</td>
<td>12</td>
<td>4</td>
<td>7</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$O_4$</td>
<td>20</td>
<td>5</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v_j$</td>
<td>6</td>
<td>1</td>
<td>-2</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

As the number of independent allocations are equal to $m+n-1$, we check the optimality.

Since all $\Delta_{ij} \geq 0$, the solution is optimal and an alternate solution exists as $\Delta_{14} = 0$. Therefore, the optimum allocation is given by

$$X_{11} = 40, X_{12} = 30, X_{22} = 5, X_{23} = 50, X_{31} = 25, X_{34} = 45, X_{41} = 20.$$  

The optimum transportation cost is:

$$= 6 \times 40 + 1 \times 30 + 5 \times 5 + 2 \times 50 + 10 \times 25 + 7 \times 45 + 0 \times 20$$

$$= Rs\ 960$$

### Example 3.13

A product is produced by 4 factories $F_1, F_2, F_3$ and $F_4$. Their unit production cost are Rs 2, 3, 1 and 5 respectively. Production capacity of the factories are 50, 70, 40 and 50 units respectively. The product is supplied to 4 stores $S_1, S_2, S_3$ and $S_4$, the requirements of which are 25, 35, 105 and 20 respectively.

Find the transportation plan such that the total production and transportation cost is minimum. Unit costs of transportation are given as follow.
NOTES

Solution We form the transportation table which consists of both production and transportation costs.

<table>
<thead>
<tr>
<th>Factory</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
<th>( S_4 )</th>
<th>( S_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 )</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>( F_2 )</td>
<td>10</td>
<td>8</td>
<td>7</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>( F_3 )</td>
<td>13</td>
<td>3</td>
<td>9</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>( F_4 )</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Total capacity = 200 units
Total demand = 185 units

Therefore, \( \sum a_i > \sum b_j \). Hence, the problem is unbalanced. We convert it into a balanced one by adding a dummy store \( S_5 \) with cost 0 and the excess supply is given as the rim requirement to this store, namely (200–185) units.

The initial basic feasible solution is obtained by least cost method. We get the solution containing 8 non-negative independent allocations equals to \( m + n - 1 \). So the solution is a non-degenerate solution.

The total transportation cost
\[= 4 \times 25 + 6 \times 5 + 8 \times 20 + 10 \times 50 + 8 \times 20 + 4 \times 30 + 13 \times 55 + 0 \times 15 = \text{Rs 1525} \]

To find the optimal solution We apply MODI method to the previous table as it has \( m + n - 1 \) independent non-negative allocations.
The solution is not optimum as the cell (4, 5) is having a negative net evaluation value, i.e., $\Delta_{45} = -3$. We draw a closed path from this cell and have a modified allocation by adding and subtracting the allocation min $(35, 20) = 20$. This modified allocation is given in the following table.

I Iteration Table

$$\begin{array}{cccccc}
 & S_1 & S_2 & S_3 & S_4 & S_5 \\
F_1 & 25 & 5 & 6 & 8 & 0 \\
F_2 & 6 & 7 & 3 & 10 & \circ \ \\
F_3 & 2 & 12 & 30 & 4 & \circ \ \\
F_4 & 9 & 0 & 11 & 15 & \circ \\
v_j & 4 & 6 & 8 & 0 & \circ \ \\
\end{array}$$

Since all the values of $\Delta_{ij} P 0$ the solution is optimum, but an alternate solution exists.

The optimum solution or the transportation plan is given by

- $X_{11} = 25$ units
- $X_{32} = 30$ units
- $X_{12} = 5$ units
- $X_{43} = 15$ units
- $X_{13} = 20$ units
- $X_{44} = 20$ units
- $X_{23} = 70$ units
- $X_{45} = 15$ units

This is the surplus capacity that are not transported which are manufactured in factory $F_4$. The optimum production with transportation cost

$$\text{Cost} = 4 \times 25 + 6 \times 5 + 8 \times 20 + 10 \times 70 + 4 \times 30 + 8 e 20$$

+ $13 \times 15 + 0 \times 15 = \text{Rs 1465}$

**Maximization case in transportation problem** In this, the objective is to maximize the total profit for which the profit matrix is given. For this, first we have to convert the maximization problem into minimization by subtracting all the
elements from the highest element in the given transportation table. This modified minimization problem can be solved in the usual manner.

**Example 3.14** There are three factories A, B and C which supply goods to four dealers D₁, D₂, D₃ and D₄. The production capacities of these factories are 1000, 700 and 900 units per month respectively. The requirements from the dealers are 900, 800, 500 and 400 units per month respectively. The per unit return (excluding transportation cost) are Rs 8, Rs 7 and Rs 9 at the three factories. The following table gives the unit transportation costs from the factories to the dealers.

<table>
<thead>
<tr>
<th></th>
<th>D₁</th>
<th>D₂</th>
<th>D₃</th>
<th>D₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Determine the optimum solution to maximize the total returns.

**Solution** Profit = return – transportation cost. With this we form a transportation table with profit.

<table>
<thead>
<tr>
<th></th>
<th>D₁</th>
<th>D₂</th>
<th>D₃</th>
<th>D₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8–2=6</td>
<td>8–2=6</td>
<td>8–2=6</td>
<td>8–4=4</td>
</tr>
<tr>
<td>B</td>
<td>7–3=4</td>
<td>7–5=2</td>
<td>7–3=4</td>
<td>7–2=5</td>
</tr>
<tr>
<td>C</td>
<td>9–4=5</td>
<td>9–3=6</td>
<td>9–2=7</td>
<td>9–1=8</td>
</tr>
</tbody>
</table>

Profit matrix

<table>
<thead>
<tr>
<th></th>
<th>D₁</th>
<th>D₂</th>
<th>D₃</th>
<th>D₄</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>4</td>
<td>1000</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>700</td>
</tr>
<tr>
<td>C</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>900</td>
</tr>
<tr>
<td>Requirement</td>
<td>900</td>
<td>800</td>
<td>500</td>
<td>400</td>
<td>2600</td>
</tr>
</tbody>
</table>

This profit matrix is converted into its equivalent loss matrix by subtracting all the elements from the highest element, namely 8. Hence, we have the following loss matrix.

<table>
<thead>
<tr>
<th></th>
<th>D₁</th>
<th>D₂</th>
<th>D₃</th>
<th>D₄</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>1000</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>700</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>900</td>
</tr>
<tr>
<td>Requirement</td>
<td>900</td>
<td>800</td>
<td>500</td>
<td>400</td>
<td>2600</td>
</tr>
</tbody>
</table>

We use VAM to get the initial basic feasible solution.
The initial solution is given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>Capacity</th>
<th>$P_1$</th>
<th>$P_{II}$</th>
<th>$P_{III}$</th>
<th>$P_{IV}$</th>
<th>$P_V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>200</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$B$</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>←</td>
</tr>
<tr>
<td>$C$</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>900</td>
<td>1</td>
<td>1</td>
<td>←</td>
<td>←</td>
<td>←</td>
</tr>
</tbody>
</table>

Demands
900 800 500 400 2600

The initial solution is given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>1000</td>
</tr>
<tr>
<td>$B$</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>700</td>
</tr>
<tr>
<td>$C$</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>900</td>
</tr>
</tbody>
</table>

Demands
900 800 500 400

Since the number of allocated cell = 5 < $m + n - 1 = 6$, the solution is degenerate. To resolve this degeneracy we add an empty cell (1, 3) by allocating a non-negative quantity $\epsilon$.

This cell is the least cost cell and is of independent position. The initial basic feasible solution is given as follows:

Initial Table

Number of allocations = 6 = $m + n - 1$ and the 6 allocations are in independent position. Hence, we can perform the optimality test using MODI method.
Since all the net evaluation $\Delta_{ji} > 0$ are non-negative the initial solution is optimum. 
The optimum distribution is 
A $\rightarrow$ D$_1$ = 200 units 
A $\rightarrow$ D$_2$ = 200 units 
A $\rightarrow$ D$_3$ = $\epsilon$ units 
B $\rightarrow$ D$_1$ = 700 units 
C $\rightarrow$ D$_3$ = 500 units 
A $\rightarrow$ D$_4$ = 400 units 
Total profit or the Max. return = $200 \times 6 + 6 \times 800 + 4 \times 700 + 7 \times 500$ 
+ $8 \times 400$ = Rs 15,500.

**Example 3.15** Solve the following transportation problem to maximize the profit.

<table>
<thead>
<tr>
<th>Source</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75</td>
<td>39</td>
<td>48</td>
<td>57</td>
<td>23</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>48</td>
<td>64</td>
<td>9</td>
<td>44</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>50</td>
<td>24</td>
<td>30</td>
<td>33</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Destination</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>Supply</td>
<td>23</td>
<td>31</td>
<td>16</td>
<td>30</td>
<td>100</td>
</tr>
</tbody>
</table>

Since $\Sigma a_i = \Sigma b_j$ there exists a feasible solution and is obtained by VAM.
The initial basic feasible solution is given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>Supply</th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_3 )</th>
<th>( P_4 )</th>
<th>( P_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75</td>
<td>39</td>
<td>48</td>
<td>57</td>
<td>23</td>
<td>9</td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>39</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>58</td>
<td>54</td>
<td>94</td>
<td>44</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>39</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>50</td>
<td>14</td>
<td>24</td>
<td>30</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Demand: 23 31 16 30

\( P_1 \): 10 9 24 ↑ 21
\( P_2 \): 10 9 18 ↑ 21
\( P_3 \): 65 ↑ 9 18 48
\( P_4 \): 9 9 48 ↑
\( P_5 \): 9 9 18

As the number of independent allocated cells = 6 = \( m + n - 1 \) the solution is non-degenerate solution.

**Optimality test using MODI method**

Since all the net evaluation \( \Delta_j > 0 \) the solution is optimum and unique. The optimum solution is given by

\[ X_{12} = 23; X_{21} = 6; X_{24} = 30; X_{31} = 17; X_{33} = 16 \]

The optimum profit = 23 \( \times \) 51 + 6 \( \times \) 80 + 42 \( \times \) 8 + 81 \( \times \) 30 + 90 \( \times \) 17 + 66 \( \times \) 16 = Rs 7005
3.6 ASSIGNMENT PROBLEMS

Suppose there are \( n \) jobs to be performed and \( n \) persons are available for doing these jobs. Assume that each person can do one job at a time, though with varying degrees of efficiency. Let \( c_{ij} \) be the cost if the \( i^{th} \) person is assigned to the \( j^{th} \) job. The problem is to find an assignment (which job should be assigned to which person on a one-one basis) so that the total cost of performing all the jobs is minimum. Problems of this type are known as assignment problems.

The assignment problem can be stated in the form of a non cost matrix \([c_{ij}]\) of real numbers as given in the following table.

```
<table>
<thead>
<tr>
<th>Jobs</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>( c_{11} )</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>( c_{21} )</td>
<td>...</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>( c_{31} )</td>
<td>...</td>
</tr>
</tbody>
</table>
```

**Mathematical Formulation of the Assignment Problem**

Mathematically, the assignment problem can be stated as

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \quad i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, n
\]

Subject to the restrictions

\[
x_{ij} = \begin{cases} 
1 & \text{if the } i^{th} \text{ person is assigned } j^{th} \text{ job} \\
0 & \text{if not} 
\end{cases}
\]

\[
\sum_{j=1}^{n} x_{ij} = 1 \quad (\text{one job is done by the } i^{th} \text{ person})
\]

and \[
\sum_{i=1}^{n} x_{ij} = 1 \quad (\text{only one person should be assigned the } j^{th} \text{ job})
\]

Where \( x_{ij} \) denotes that the \( j^{th} \) job is to be assigned to the \( i^{th} \) person.
### Difference between Transportation Problem and Assignment Problem

<table>
<thead>
<tr>
<th>Transportation problem</th>
<th>Assignment problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. No. of sources and number of destinations need not be equal. Hence, the cost matrix is not necessarily a square matrix.</td>
<td>Since assignment is done on one-to-one basis, the number of sources and the number of destinations are equal. Hence, the cost matrix must be a square matrix.</td>
</tr>
<tr>
<td>2. $x_{ij}$, the quantity to be transported from $i$th origin to $j$th destination can take any possible positive values, and satisfies the rim requirements.</td>
<td>$x_{ij}$, the $j$th job is to be assigned to the $i$th person and can take either the value 1 or 0.</td>
</tr>
<tr>
<td>3. The capacity and the requirement value is equal to $a_i$ and $b_j$ for the $i$th source and $j$th destinations $(i=1, 2m, j=1, 2… n)$</td>
<td>The capacity and the requirement value is exactly 1 i.e., for each source of each destination the capacity and the requirement value is exactly 1.</td>
</tr>
<tr>
<td>4. The problem is unbalanced if the total supply and the total demand are not equal.</td>
<td>The problem is unbalanced if the cost matrix is not a square matrix.</td>
</tr>
</tbody>
</table>

### Hungarian Method Procedure

The solution of an assignment problem can be arrived using the Hungarian Method. The steps involved in this method are as follows.

**Step 1** Prepare a cost matrix. If the cost matrix is not a square matrix then add a dummy row (column) with zero cost element.

**Step 2** Subtract the minimum element in each row from all the elements of the respective rows.

**Step 3** Further modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus, obtain the modified matrix.

**Step 4** Then, draw minimum number of horizontal and vertical lines to cover all zeroes in the resulting matrix. Let the minimum number of lines be $N$. Now, there are two possible cases.

- **Case i** If $N = n$, where $n$ is the order of matrix, then an optimal assignment can be made. So make the assignment to get the required solution.
- **Case ii** If $N < n$, then proceed to step 5.

**Step 5** Determine the smallest uncovered element in the matrix (element not covered by $N$ lines). Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus, the second modified matrix is obtained.

**Step 6** Repeat step (3) and (4) until we get the case (i) of Step 4.

**Step 7** (To make zero assignment) Examine the rows successively until a row-wise exactly single zero is found. Circle (O) this zero to make the assignment. Then mark a cross (∗) over all zeroes if lying in the column of the circled zero, showing...
that they can not be considered for future assignment. Continue in this manner until all the zeroes have been examined. Repeat the same procedure for the column also.

**Step 8** Repeat the step 6 successively until one of the following situation arises

(i) If no unmarked zero is left, then the process ends.

(ii) If there lies more than one of the unmarked zero in any column or row, then circle one of the unmarked zeroes arbitrarily and mark a cross in the cells of remaining zeroes in its row or column. Repeat the process until no unmarked zero is left in the matrix.

**Step 9** Thus, exactly one marked circled zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked circled zeroes will give the optimal assignment.

**Example 3.16** Using the following cost matrix, determine (a) the optimal job assignment (b) the cost of assignments.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Job</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>10</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>C</td>
<td>9</td>
<td>7</td>
<td>8</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>D</td>
<td>7</td>
<td>5</td>
<td>6</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>10</td>
<td>9</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

**Solution** Select the smallest element in each row and subtract this smallest element from all the elements in its row.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>8</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>C</td>
<td>7</td>
<td>5</td>
<td>6</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>D</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

Subtract the minimum element from each column and subtract this element from all the elements in its column. With this, we get the first modified matrix.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

In this modified matrix, we draw the minimum number of lines to cover all zero (horizontal or vertical).
The number of lines drawn to cover all zeroes is $4 = N$.
The order of matrix is $n = 5$.
Hence, $N < n$.

Now we get the second modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and add it to the element at the point of intersection of lines.

The number of lines drawn to cover all zeroes = $N = 5$.
The order of matrix is $n = 5$.
Hence, $N = n$. Now, we determine the optimum assignment.

**Assignment**

The first row contains more than one zero. So proceed to the second row. It has exactly one zero. The corresponding cell is $(B, 4)$. Circle this zero, thus making an assignment. Mark ($\times$) for all other zeroes in its column. Showing that they cannot be used for making other assignments. Now, row 5 has a single zero in the cell $(E, 3)$. Make an assignment in this cell and cross the second zero in the third column.
Now, row 1 has a single zero in the column 2, i.e., in the cell \((A, 2)\).
Make an assignment in this cell and cross the other zeroes in the second column. This leads to a single zero in column 1 of the cell \((D, 1)\), second. Make an assignment in this cell and cross the other zeroes in the fourth row. Finally, we have a single zero left in the third row making an assignment in the cell \((C, 5)\). Thus, we have the following assignment.

Optimal assignment and optimum cost of assignment.

<table>
<thead>
<tr>
<th>Job</th>
<th>Mechanic</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>D</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>E</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>B</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>C</td>
<td>4</td>
</tr>
</tbody>
</table>

Therefore \(1 \rightarrow D, 2 \rightarrow A, 3 \rightarrow E, 4 \rightarrow B, 5 \rightarrow C\) with minimum cost Rs 21.

**Example 3.17** A company has 5 jobs to be done on five machines. Any job can be done on any machine. The cost of doing the jobs in different machines are given below. Assign the jobs for different machines so as to minimise the total cost.

**Solution** We form the first modified matrix by subtracting the minimum element from all the elements in the respective row and same with respective column.

$$
\begin{bmatrix}
A & B & C & D & E \\
1 & 13 & 8 & 16 & 18 & 19 \\
2 & 9 & 15 & 24 & 9 & 12 \\
3 & 12 & 0 & 4 & 4 & 4 \\
4 & 6 & 12 & 10 & 8 & 13 \\
5 & 15 & 17 & 18 & 12 & 20 \\
\end{bmatrix}
$$

Since each column has the minimum element 0. We have the first modified matrix. Now, we draw the minimum number of lines to cover all zeroes.
The number of lines drawn to cover all zeroes is \( N = 4 < \) the order of matrix is \( n = 5 \).

We find the second modified matrix by subtracting the smallest uncovered element from all the uncovered elements and add to the element which is in the point of intersection of lines.

The number of lines drawn to cover all zeroes = 5.
This is the order of matrix. Hence, we can form an assignment.

### Assignment

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>0</td>
<td>8</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>6</td>
<td>15</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>0</td>
<td>8</td>
</tr>
</tbody>
</table>

All the five jobs have been assigned to 5 different machines.

Here, the optimal assignment is.

<table>
<thead>
<tr>
<th>Job</th>
<th>Machine</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>B</td>
</tr>
<tr>
<td>2</td>
<td>E</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
</tr>
<tr>
<td>4</td>
<td>A</td>
</tr>
<tr>
<td>5</td>
<td>D</td>
</tr>
</tbody>
</table>

Minimum (Total cost) = \( 8 + 12 + 4 + 6 + 12 = \) Rs 42
**Example 3.18** Four different jobs can be done on four different machines and take down time costs are prohibitively high for change overs. The following matrix gives the cost in rupees of producing job \( i \) on machine \( j \):

<table>
<thead>
<tr>
<th>Jobs</th>
<th>( M_1 )</th>
<th>( M_2 )</th>
<th>( M_3 )</th>
<th>( M_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J_1 )</td>
<td>5</td>
<td>7</td>
<td>11</td>
<td>6</td>
</tr>
<tr>
<td>( J_2 )</td>
<td>8</td>
<td>5</td>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>( J_3 )</td>
<td>4</td>
<td>7</td>
<td>10</td>
<td>7</td>
</tr>
<tr>
<td>( J_4 )</td>
<td>10</td>
<td>4</td>
<td>8</td>
<td>3</td>
</tr>
</tbody>
</table>

How should the jobs be assigned to the various machines so that the total cost is minimized?

**Solution** We form a first modified matrix by subtracting the least element in the respective row and respective column.

\[
\begin{bmatrix}
J_1 & 0 & 2 & 6 & 1 \\
J_2 & 3 & 0 & 4 & 1 \\
J_3 & 0 & 3 & 6 & 3 \\
J_4 & 7 & 1 & 5 & 0 \\
\end{bmatrix}
\]

Since the third column has no zero element, we subtract the smallest element 4 from all the elements.

\[
\begin{bmatrix}
M_1 & M_2 & M_3 & M_4 \\
J_1 & 0 & 2 & 2 & 1 \\
J_2 & 3 & 0 & 0 & 1 \\
J_3 & 0 & 3 & 2 & 3 \\
J_4 & 7 & 1 & 1 & 0 \\
\end{bmatrix}
\]

Now we draw minimum number of lines to cover all zeroes.

The number of lines drawn to cover all zeroes = 3 which is less than the order of matrix \( 4 \).

Hence, we form the second modified matrix, by subtracting the smallest uncovered element from all the uncovered elements and adding to the element which is in the point of intersection of lines.
Transportation Problems

NOTES

Hence, we can make an assignment.

Since no rows and no columns have single zero, we have different assignment (Multiple solution).

**Optimal assignment**

<table>
<thead>
<tr>
<th>Job</th>
<th>Machine</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_1$</td>
<td>$M_4$</td>
</tr>
<tr>
<td>$J_2$</td>
<td>$M_2$</td>
</tr>
<tr>
<td>$J_3$</td>
<td>$M_1$</td>
</tr>
<tr>
<td>$J_4$</td>
<td>$M_3$</td>
</tr>
</tbody>
</table>

Minimum (Total cost)  
$6+5+4+8 = Rs 23$

**Alternate solution**

$J_1 \rightarrow M_4; J_2 \rightarrow M_2 = J_3 \rightarrow M_1; J_4 \rightarrow M_3.$

Minimum (Total cost) $5+5+10+3 = Rs 23$

**Example 3.19** Solve the following assignment problem in order to minimize the total cost. The following cost matrix given gives the assignment cost when different operators are assigned to various machines.
NOTES

Solution We form the first modified matrix by subtracting the least element from all the elements in the respective rows and then in the respective columns.

\[
\begin{bmatrix}
5 & 0 & 8 & 10 & 11 \\
0 & 6 & 15 & 0 & 3 \\
8 & 5 & 0 & 0 & 0 \\
0 & 6 & 4 & 2 & 7 \\
3 & 5 & 6 & 0 & 8
\end{bmatrix}
\]

Since each column has the minimum element 0, the first modified matrix is obtained. We draw the minimum number of lines to cover all zeroes.

The number of lines drawn to cover all zeroes = 4 = the order of matrix = 5. Hence, we form the second modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding to the element which is in the point of intersection of lines.

\[
\begin{bmatrix}
5 & 0 & 8 & 10 & 11 \\
0 & 6 & 12 & 0 & 1 \\
11 & 8 & 0 & 3 & 0 \\
6 & 1 & 2 & 4 \\
3 & 5 & 3 & 0 & 5
\end{bmatrix}
\]

\(N = 5\), i.e., the number of lines drawn to cover all zeroes = order of matrix. Hence, we can make assignment.

The optimum assignment is

\[
\begin{array}{c|c}
\text{Operators} & \text{Machines} \\
\hline
I & D \\
II & A \\
III & C \\
IV & E \\
V & B \\
\end{array}
\]

The optimum cost is given by

\[25 + 25 + 22 + 24 + 26 = \text{Rs } 122\]
Unbalanced Assignment problem

Any assignment problem is said to be unbalanced if the cost matrix is not a square matrix, i.e., the number of rows and the number of columns are not equal. To make it balanced, we add a dummy row or a dummy column with all the entries as zero.

Example 3.20 There are four jobs to be assigned to the machines. Only one job could be assigned to one machine. The amount of time in hours required for the jobs in a machine are given in the following matrix.

<table>
<thead>
<tr>
<th>Job</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>3</td>
<td>6</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>12</td>
<td>11</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>9</td>
<td>6</td>
</tr>
</tbody>
</table>

Find an optimum assignment of jobs to the machines to minimize the total processing time and also, find for which machine no job is assigned. What is the total processing time to complete all the jobs?

Solution Since the cost matrix is not a square matrix, the problem is unbalanced. We add a dummy job 5 with corresponding entries zero.

Modified matrix

\[ \begin{bmatrix}
1 & 4 & 3 & 6 & 2 & 7 \\
2 & 10 & 12 & 11 & 14 & 16 \\
3 & 4 & 3 & 2 & 1 & 5 \\
4 & 8 & 7 & 6 & 9 & 6 \\
5 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

We subtract the smallest element from all the elements in the respective row.

\[ \begin{bmatrix}
1 & 2 & 1 & 4 & 0 & 5 \\
2 & 0 & 2 & 1 & 4 & 6 \\
3 & 3 & 2 & 1 & 0 & 4 \\
4 & 2 & 1 & 0 & 3 & 0 \\
5 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

Since each column has minimum element as zero, we draw the minimum number of lines to cover all zeroes.

Check Your Progress

5. When does degeneracy occur in transportation problems?
6. When do you say that a TP is unbalanced?
7. Cite one difference between a transportation problem and an assignment problem.
8. What is an unbalanced assignment problem?
The number of lines to cover all zeroes = 4 < the order of matrix. We form the second modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding to the element at the point of intersection of lines.

Here the number of lines drawn to cover all zeroes = 5 = Order of matrix. Therefore, we can make the assignment

Optimum assignment

<table>
<thead>
<tr>
<th>Job</th>
<th>Machine</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>C</td>
</tr>
<tr>
<td>3</td>
<td>D</td>
</tr>
<tr>
<td>4</td>
<td>E</td>
</tr>
</tbody>
</table>

For machine E no job is assigned.

Optimum (minimum) cost = \[4 + 11 + 1 + 6 = 22.\]

Example 3.21 A company has 4 machines to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given as follows. Determine the job assignments which will minimize the total cost.
**Solution** Since the cost matrix is not a square matrix we add a dummy row $D$ with all the elements 0.

\[
\begin{array}{cccc}
W & X & Y & Z \\
A & 18 & 24 & 28 & 32 \\
B & 8 & 13 & 17 & 18 \\
C & 10 & 15 & 19 & 22 \\
D & 0 & 0 & 0 & 0 \\
\end{array}
\]

Subtract the minimum element in each row from all the elements in its row.

\[
\begin{array}{cccc}
W & X & Y & Z \\
A & 0 & 6 & 10 & 14 \\
B & 0 & 5 & 9 & 10 \\
C & 0 & 5 & 9 & 12 \\
D & 0 & 0 & 0 & 0 \\
\end{array}
\]

Since each column has minimum element we draw minimum number of lines to cover all zeroes.

\[
\begin{array}{cccc}
W & X & Y & Z \\
A & 0 & 6 & 10 & 14 \\
B & 0 & 5 & 9 & 10 \\
C & 0 & 5 & 9 & 12 \\
D & 0 & 0 & 0 & 0 \\
\end{array}
\]

∴ the number of lines drawn to cover all zeroes = 2 < the order of matrix, we form a second modified matrix.

\[
\begin{array}{cccc}
W & X & Y & Z \\
A & 0 & 1 & 5 & 9 \\
B & 0 & 0 & 4 & 5 \\
C & 0 & 0 & 4 & 7 \\
D & 5 & 0 & 0 & 0 \\
\end{array}
\]

Here, $N = 3 < n = 4$.

Again, we subtract the smallest uncovered element from all the uncovered elements and add to the element at the point of intersection

\[
\begin{array}{cccc}
W & X & Y & Z \\
A & 0 & 1 & 1 & 4 \\
B & 0 & 0 & 0 & 1 \\
C & 0 & 0 & 0 & 3 \\
D & 9 & 4 & 0 & 0 \\
\end{array}
\]

Here, $N = 4 = n$. Hence, we make an assignment.
Assignment

Since $D$ is a dummy job, machine $Z$ has assigned no job.

Therefore, optimum cost = 18 + 13 + 19 = Rs 50

Maximization in Assignment problem

In this, the objective is to maximize the profit. To solve this, we first convert the given profit matrix into the loss matrix by subtracting all the elements from the highest element of the given profit matrix. For this converted loss matrix, we apply the steps in Hungarian method to get the optimum assignment.

Example 3.22 The owner of a small machine shop has four mechanics available to assign jobs for the day. Five jobs are offered with expected profit for each mechanic on each job which are as follows:

Find by using the assignment method, the assignment of mechanics to the job that will result in a maximum profit. Which job should be declined?

Solution The given profit matrix is not a square matrix as the number of jobs is not equal to the number of mechanics. Hence, we introduce a dummy mechanic 5 with all the elements 0.

Now we convert this profit matrix into loss matrix by subtracting all the elements from the highest element 111.
We subtract the smallest element from all the elements in the respective row.

\[
\begin{array}{ccccc}
A & B & C & D & E \\
1 & 39 & 30 & 58 & 0 & 19 \\
2 & 6 & 0 & 23 & 5 & 18 \\
3 & 17 & 19 & 0 & 33 & 23 \\
4 & 32 & 23 & 0 & 3 & 0 \\
5 & 0 & 7 & 7 & 0 & 0 \\
\end{array}
\]

Since each column has the minimum element as zero, we draw the minimum number of lines to cover all zeroes.

\[
\begin{array}{ccccc}
A & B & C & D & E \\
1 & 39 & 30 & 58 & 0 & 19 \\
2 & 6 & 0 & 23 & 5 & 18 \\
3 & 17 & 19 & 0 & 33 & 23 \\
4 & 32 & 23 & 0 & 3 & 0 \\
5 & 0 & 7 & 7 & 0 & 0 \\
\end{array}
\]

Here, the number of lines drawn to cover all zeroes = \(N = 4\) is less than the order of matrix.

We form the second modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding to the element which is at the point of intersection of lines.

\[
\begin{array}{ccccc}
A & B & C & D & E \\
1 & 39 & 30 & 58 & 0 & 19 \\
2 & 6 & 0 & 23 & 5 & 18 \\
3 & 17 & 19 & 0 & 33 & 23 \\
4 & 32 & 23 & 0 & 3 & 0 \\
5 & 0 & 7 & 7 & 0 & 0 \\
\end{array}
\]

Here, \(N = 5 = n\) (the order of matrix).

We make the assignment.
The optimum assignment is

<table>
<thead>
<tr>
<th>Job</th>
<th>Mechanic</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
</tr>
<tr>
<td>E</td>
<td>4</td>
</tr>
</tbody>
</table>

Since the fifth mechanic is a dummy, job $A$ is assigned to the fifth mechanic, this job is declined.

The maximum profit is given by $84 + 111 + 101 + 80 = \text{Rs } 376$

**Example 3.23** A marketing manager has 5 salesmen and there are 5 sales districts. Considering the capabilities of the salesmen and the nature of districts, the estimates made by the marketing manager for the sales per month (in 1000 rupees) for each salesman in each district would be as follows.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>32</td>
<td>38</td>
<td>40</td>
<td>28</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>24</td>
<td>28</td>
<td>21</td>
<td>36</td>
</tr>
<tr>
<td>3</td>
<td>41</td>
<td>27</td>
<td>33</td>
<td>30</td>
<td>37</td>
</tr>
<tr>
<td>4</td>
<td>22</td>
<td>38</td>
<td>41</td>
<td>36</td>
<td>36</td>
</tr>
<tr>
<td>5</td>
<td>29</td>
<td>33</td>
<td>40</td>
<td>35</td>
<td>39</td>
</tr>
</tbody>
</table>

Find the assignment of salesmen to the districts that will result in the maximum sales.

**Solution** We are given the profit matrix. To maximize the profit, first we convert it into loss matrix which can be minimized. To convert it into loss matrix, we subtract all the elements from the highest element 41. Subtract the smallest element from all the elements in the respective rows and columns to get the first modified matrix.

<table>
<thead>
<tr>
<th>Loss Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
</tbody>
</table>
We now draw the minimum number of lines to cover all zeroes.

\[
\begin{array}{cccccc}
A & B & C & D & E \\
1 & 8 & 0 & 0 & 7 & 0 \\
2 & 0 & 14 & 12 & 14 & 4 \\
3 & 0 & 12 & 8 & 6 & 4 \\
4 & 9 & 1 & 0 & 0 & 5 \\
5 & 11 & 5 & 0 & 0 & 1 \\
\end{array}
\]

\(N = 4 < n = 5\)

We subtract the smallest uncovered element from the remaining uncovered elements and add to the elements at the point of intersection of lines to get the second modified matrix.

\[
\begin{array}{cccccc}
A & B & C & D & E \\
1 & 9 & 0 & 1 & 8 & 0 \\
2 & 0 & 13 & 12 & 4 & 3 \\
3 & 0 & 11 & 8 & 6 & 3 \\
4 & 9 & 0 & 0 & 0 & 4 \\
5 & 11 & 4 & 0 & 0 & 0 \\
\end{array}
\]

Again, \(N = 4 < n = 5\). Repeat the previous step.

\[
\begin{array}{cccccc}
A & B & C & D & E \\
1 & 4 & 3 & 6 & 2 & 7 \\
2 & 10 & 12 & 11 & 14 & 16 \\
3 & 4 & 3 & 2 & 1 & 5 \\
4 & 8 & 7 & 6 & 9 & 6 \\
5 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\(N = 5 = n = 5\). Hence, we make the assignment.
NOTES

Assignment

\[
\begin{array}{ccccc}
 & A & B & C & D & E \\
1 & 12 & 0 & 1 & 8 & 0 \\
2 & 0 & 10 & 9 & 1 & 0 \\
3 & 0 & 8 & 5 & 3 & 0 \\
4 & 12 & 0 & 0 & 0 & 4 \\
5 & 14 & 4 & 0 & 0 & 0 \\
\end{array}
\]

Since no row or column has single zero, we get a multiple solution.

(i) The optimum assignment is:

1 → B, 2 → A, 3 → E, 4 → C, 5 → D

With maximum profit (38|40|37|41|35): Rs 191

(ii) The optimum assignment is:

1 → B, 2 → A, 3 → E, 4 → D, 5 → C

Maximum profit (38|40|37|36|40): Rs 191

3.7 TRAVELLING SALESMAN PROBLEM

Assume a salesman has to visit \( n \) cities. He wishes to start from a particular city, visit each city once and then return to his starting point. His objective is to select the sequence in which the cities are visited in such a way that his total travelling time is minimized.

To visit 2 cities (A and B, there is no choice. To visit 3 cities we have 2! possible routes. For 4 cities we have 3! possible routes. In general, to visit \( n \) cities there are \((n − 1)!\) possible routes.

Mathematical Formulation

Let \( C_{ij} \) be the distance or time or cost of going from city \( i \) to city \( j \). the decision variable \( X_{ij} \) be 1 if the salesman travels from city \( i \) to city \( j \) and otherwise 0.

The objective is to minimize the travelling time.

\[
Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}
\]

Subject to the constraints

\[
\sum_{j=1}^{n} X_{ij} = 1, \; i = 2 \ldots n.
\]

\[
\sum_{i=1}^{n} X_{ij} = 1, \; j = 2 \ldots n.
\]
and subject to the additional constraint that \( X_{ij} \) is so chosen that no city is visited twice before all the cities are completely visited.

\[ \ldots, \ldots, \ldots, \ldots, i \rightarrow j \ldots, \ldots \]

\( i \) directly to \( j \) is not permitted. Which means \( C_{ii} = \infty \), when \( i = j \).

In travelling salesman problem we cannot choose the element along the diagonal and this can be avoided by filling the diagonal with infinitely large elements.

The travelling salesman problem is very similar to the assignment problem except that in the former care, there is an additional restriction, that \( X_{ij} \) is so chosen that no city is visited twice before the tour of all the cities is completed.

**Solution** Treat the problem as an assignment problem and solve it using the same procedures. If the optimal solution of the assignment problem satisfies the additional constraint, then it is also an optimal solution of the given travelling salesman problem. If the solution to the assignment problem does not satisfy the additional restriction then after solving the problem by assignment technique we use the method of enumeration.

**Example 3.24** A travelling salesman has to visit 5 cities. He wishes to start from a particular city, visit each city once and then return to his starting point. Cost of going from one city to another is shown below. You are required to find the least cost route.

**Solution** First we solve this problem as an assignment problem.

Subtract the minimum element in each row from all the elements in its row.

\[
\begin{bmatrix}
 A & B & C & D & E \\
 A & \infty & 4 & 10 & 14 & 2 \\
 B & 12 & \infty & 6 & 10 & 4 \\
 C & 16 & 14 & \infty & 8 & 14 \\
 D & 24 & 8 & 12 & \infty & 10 \\
 E & 2 & 6 & 4 & 16 & \infty 
\end{bmatrix}
\]

Subtract the minimum element in each column from all the elements in its column.

\[
\begin{bmatrix}
 A & B & C & D & E \\
 A & \infty & 2 & 8 & 12 & 0 \\
 B & 8 & \infty & 2 & 6 & 0 \\
 C & 8 & 6 & \infty & 0 & 6 \\
 D & 16 & 0 & 4 & \infty & 2 \\
 E & 0 & 4 & 2 & 14 & \infty 
\end{bmatrix}
\]
We have the first modified matrix. Draw minimum number of lines to cover all zeroes.

\[
\begin{array}{ccccc}
A & B & C & D & E \\
A & \infty & 2 & 8 & 12 & 0 \\
B & 8 & \infty & 2 & 6 & 0 \\
C & 8 & 6 & \infty & 0 & 6 \\
D & 16 & 0 & 4 & \infty & 2 \\
E & 0 & 4 & 2 & 14 & \infty \\
\end{array}
\]

\[N = 4 \quad 0 \quad n = 5\]
Subtract the smallest uncovered element from all the uncovered elements and add to the element which is in the point of intersection of lines. Hence, we get the second modified matrix.

\[
\begin{array}{ccccc}
A & B & C & D & E \\
A & \infty & 2 & 8 & 12 & 0 \\
B & 8 & \infty & 2 & 6 & 0 \\
C & 8 & 6 & \infty & 0 & 6 \\
D & 16 & 0 & 4 & \infty & 2 \\
E & 0 & 4 & 2 & 14 & \infty \\
\end{array}
\]

\[N = 5, \quad n = 5\]
We make the assignment.

**Assignment**

\[
\begin{array}{ccccc}
A & B & C & D & E \\
A & \infty & 0 & 6 & 12 & 0 \\
B & 8 & \infty & 0 & 6 & 0 \\
C & 8 & 4 & \infty & 0 & 6 \\
D & 18 & \infty & 4 & \infty & 4 \\
E & \infty & 4 & \infty & 14 & \infty \\
\end{array}
\]

\[A \rightarrow E \rightarrow A\]

As the salesman should go from \(A\) to \(E\) and then come back to \(A\) without covering \(B, C, D\) which is contradicting the fact that no city is visited twice before all the cities are visited.

Hence, we obtain the next best solution by bringing the next minimum non-zero element, namely 4.
A → B, B → C, C → D, D → E, E → A

Since all the cities have been visited and no city is visited twice before completing the tour of all the cities, we have an optimal solution to the travelling salesman.

The least cost route is A → B → C → D → E → A.

Total cost = 4 + 6 + 8 + 10 + 2 = Rs 30

Example 3.25 A machine operator processes five types of items on his machine each week and must choose a sequence for them. The set-up cost per change depends on the items presently on the machine and the set-up to be made according to the following table.

<table>
<thead>
<tr>
<th>To item</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>∞</td>
<td>4</td>
<td>7</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>∞</td>
<td>6</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>From item</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>7</td>
<td>6</td>
<td>∞</td>
</tr>
<tr>
<td>D</td>
<td>3</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>E</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

If he processes each type of item only once in each week, how should he sequence the items on his machine in order to minimize the total set-up cost?

Solution Reduce the cost matrix and make assignments in rows and columns having single row.

Modify the matrix by subtracting the least element from all the elements in its row and also in its column.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>∞</td>
<td>1</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>∞</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>1</td>
<td>∞</td>
<td>2</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>∞</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
NOTES

Here, \( N = 4 \) \( n = 5 \), i.e., \( N < n \).

Subtract the smallest uncovered element from all the uncovered elements and add to the element which is at the point of intersection of lines and get the reduced second modified matrix.

\[
\begin{array}{cccccc}
A & B & C & D & E \\
\infty & 1 & 3 & 0 & 1 \\
1 & \infty & 2 & 0 & 1 \\
2 & 1 & \infty & 2 & 0 \\
0 & 0 & 3 & \infty & 4 \\
0 & 0 & 0 & 4 & \infty \\
\end{array}
\]

\( N = 5 = n = 5 \). We make the assignment.

**Assignment**

\[
\begin{array}{cccccc}
A & B & C & D & E \\
\infty & \infty & 2 & \infty & 1 & A \rightarrow B \\
\infty & 1 & \infty & 1 & 1 & B \rightarrow D \\
1 & 0 & \infty & 1 & \infty & C \rightarrow E \\
0 & \infty & 3 & \infty & 5 & D \rightarrow A \\
0 & 0 & 4 & \infty & \infty & E \rightarrow C \\
\end{array}
\]

We get the solution \( A \rightarrow B \rightarrow D \rightarrow A \).

This schedule provides the required solution as each item is not processed once in a week.

Hence, we make a better solution by considering the next smallest non-zero element by considering 1.

\[
\begin{array}{cccccc}
A & B & C & D & E \\
\infty & \infty & 2 & \infty & 1 \\
\infty & \infty & \infty & 0 & 1 \\
1 & 0 & \infty & 1 & \infty \\
0 & \infty & 3 & \infty & 5 \\
0 & 0 & 4 & \infty & \infty \\
\end{array}
\]

\( A \rightarrow E, E \rightarrow C, C \rightarrow B, B \rightarrow D, D \rightarrow A, \)

i.e., \( A \rightarrow E \rightarrow C \rightarrow B \rightarrow D \rightarrow A \).

The total set-up cost comes to Rs 21.
3.8 SUMMARY

In this unit, you have learned that in order to achieve the minimum cost in transportation problems, we must know the costs that result from transporting one unit of a commodity from various origins to various destinations. The optimal solution is a feasible solution which minimizes the total cost. The unit has described the MODI method for determining the optimal solution. You have also learned that degeneracy may occur either at the initial stage or at any subsequent iteration. The unit has explained that a transportation problem is said to be unbalanced if the total supply is not equal to the total demand, whereas in an assignment problem the problem is unbalanced if the cost matrix is not a square matrix. In the travelling salesman problem, the objective of the salesman is to select a sequence in which the cities are visited in such a way that his total travelling time is minimized.

3.9 KEY TERMS

- **Transportation problem**: A subclasses of LPPs in which the objective is to transport various quantities of a single homogeneous commodity that are initially stored at various origins, to different destinations in such a way that the transportation cost is minimum.

- **Basic feasible solution**: A feasible solution in which the number of non-negative allocations is equal to \( m \times n \) where \( m \) is the number of rows and \( n \) the number of columns in a transportation table.

- **Non-degenerate basic feasible solution**: Any feasible solution to a transportation problem containing \( m \) origins and \( n \) destinations is said to be non-degenerate, if it contains \( m \times n \) occupied cells and each allocation is in independent positions.

- **Degenerate basic feasible solution**: If a basic feasible solution contains less than \( m \times n \) non negative allocations, it is said to be degenerate.

- **Optimal solution**: A feasible solution (not necessarily basic) which minimizes the total cost. The solution of a transportation problem can be obtained in two stages, namely initial solution and optimum solution.

3.10 ANSWERS TO ‘CHECK YOUR PROGRESS’

1. The transportation problem is one of the subclasses of LPPs in which the objective is to transport various quantities of a single homogeneous commodity that are initially stored at various origins to different destinations in such a way that the transportation cost is minimum.

2. Any feasible solution to a transportation problem containing \( m \) origins and \( n \) destinations is said to be non-degenerate, if it contains \( m \times n \) occupied cells and each allocation is in independent positions.
3. Initial solution can be obtained by using any one of the three methods, viz,
   (i) North west corner rule (NWCR)
   (ii) Least cost method or matrix minima method
   (iii) Vogel's approximation method (VAM)
4. VAM is preferred over the other two methods since the initial basic feasible solution obtained by this method is either optimal or very close to the optimal solution.
5. In a TP, if the number of non-negative independent allocations is less than \( m + n - 1 \), where \( m \) is the number of origins (rows) and \( n \) is the number of destinations (columns) there exists a degeneracy. This may occur either at the initial stage or at subsequent iteration.
6. The given TP is said to be unbalanced if \( S_{ai} \neq S_{bj} \) i.e., if the total supply is not equal to the total demand.

7. \( \text{Transportation problem} \hspace{1cm} \text{Assignment problem} \)
   No. of sources and number of destinations need not be equal. Hence, the cost matrix is not necessarily a square matrix.
   Since assignment is done on one basis, the number of sources and the number of destinations are equal. Hence, the cost matrix must be a square matrix.

8. Any assignment problem is said to be unbalanced if the cost matrix is not a square matrix, i.e., the number of rows and the number of columns are not equal. To make it balanced, we add a dummy row or a dummy column with all the entries as zero.

### 3.11 QUESTIONS AND EXERCISES

**Short-Answer Questions**

1. What do you understand by transportation model?
2. Define feasible solution, basic solution, non-degenerate solution, optimal solution in a transportation problem.
3. Give the mathematical formulation of a TP.
4. Give the mathematical formulation of an assignment problem.

**Long-Answer Questions**

1. Explain the following briefly with examples:
   I. North West Corner Rule
   II. Least Cost Method
   III. Vogel’s Approximation Method
2. Explain degeneracy in a TP. Describe a method to resolve it.
3. What do you mean by an unbalanced TP? Explain how to convert an unbalanced T.P. into a balanced one.
4. Explain an algorithm to solving a transportation problem.
5. Describe an assignment problem giving a suitable example.
6. Explain the difference between a transportation problem and an assignment problem.
7. How can you maximize an objective function in the assignment problem? Explain the nature of I in traveling salesman problem and give its mathematical formulation.

3.12 FURTHER READING


Decision-making is an everyday process in life. It is the major job of a manager too. A decision taken by a manager has far-reaching consequences on the business. Right decisions will have a salutary effect and wrong ones may prove to be disastrous.

Decisions may be classified into two categories, tactical and strategic. Tactical decisions are those which affect the business in the short run. Strategic decisions are those which have a long-term effect on the course of business.

These days, in every organization—large or small, the person at the top has to take the crucial decisions, knowing that certain events beyond his control may occur and make him regret the decision. He is uncertain as to whether or not these unfortunate events will happen. In such situations the best possible decision can be made with the use of statistical tools which try to minimize the degree to which the person is likely to regret the decision he takes for a particular problem.

The problem under study may be represented by a model in terms of the following elements:

(i) **The decision-maker** The decision-maker is charged with the responsibility of taking a decision; he has to select one from a set of possible courses of action.

(ii) **Acts** These are the alternative courses of action or strategies that are available to the decision-maker. The decision involves a selection among two or more alternative courses of action. The problem is to choose the best of these alternatives to achieve an objective.
(iii) **Events** These are the occurrences which affect the achievement of the objectives. They are also called states of nature or outcomes. The events constitute a mutually exclusive and exhaustive set of outcomes which describe the possible behaviour of the environment in which the decision is made. The decision-maker has no control over which event will take place and can only attach a subjective probability of occurrence of each.

(iv) **Pay-off table** It represents the economics of a problem, i.e., the revenue and costs associated with any action with a particular outcome. It is an ordered statement of profit or costs resulting under a given situation. A pay-off can be interpreted as the outcome in quantitative form if the decision-maker adopts a particular strategy under a particular state of nature.

(v) **Opportunity loss table** An opportunity loss is the loss incurred because of the failure to take the best possible action. Opportunity losses are calculated separately for each state of nature that might occur. Given the occurrence of a specific state of nature, we can determine the best possible act. For a given state of nature, the opportunity loss of an act is the difference between the pay-off of that act and the pay-off for the best act that could have been selected.

### 4.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Analyse the nature of a decision-making environment
- Understand the various choices available for decision-making under uncertainty
- Describe how decisions are taken under risk
- Describe the procedure of decision tree analysis
- Understand the importance and applications of integer programming
- Describe the characteristic features of dynamic programming problems

### 4.2 DECISION-MAKING ENVIRONMENT

In any decision problem, the decision-maker is concerned with choosing from among the available alternative courses of action, the one that yields the best result. If the consequences of each choice are known with certainty, the decision-maker can easily make decisions. But in most of real life problems, the decision-maker has to deal with situations where uncertainty of the outcomes prevail.

The decision-making problems can be discussed under the following heads on the basis of their environments:

1. Decision-making under certainty
2. Decision-making under uncertainty
3. Decision-making under risk
4. Decision-making under conflict
4.2.1 Decision-Making under Certainty

In this case, the decision-maker knows with certainty the consequences of every alternative or decision choice. The decision-maker presumes that only one state of nature is relevant for his purpose. He identifies this state of nature, takes it for granted and presumes complete knowledge as to its occurrence.

4.2.2 Decision-Making under Uncertainty

When the decision-maker faces multiple states of nature but he has no means to arrive at probability values to the likelihood of occurrence of these states of nature, the problem is a decision problem under uncertainty. Such situations arise when a new product is introduced in the market or a new plant is set up. In business, there are many problems of this ‘nature’. Here, the choice of decision largely depends on the personality of the decision-maker.

The following choices are available before the decision-maker in situations of uncertainty.

(a) Maximax Criterion
(b) Minimax Criterion
(c) Maximin Criterion
(d) Laplace Criterion (Criterion of equally likelihood)
(e) Hurwicz Alpha Criterion (Criterion of Realism)

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(a) Maximax Criterion
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(e) Hurwicz Alpha Criterion (Criterion of Realism)

**The maximax decision criterion (criterion of optimism)** The term ‘maximax’ is an abbreviation of the phrase maximum of the maximums, and an adventurous and aggressive decision-maker may choose to take the action that would result in the maximum pay-off possible. Suppose for each action there are three possible pay-offs corresponding to three states of nature as given in the following decision matrix:

<table>
<thead>
<tr>
<th>States of nature</th>
<th>Decisions</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A₁</td>
<td>A₂</td>
<td>A₃</td>
<td></td>
</tr>
<tr>
<td>S₁</td>
<td>220</td>
<td>180</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>S₂</td>
<td>160</td>
<td>190</td>
<td>180</td>
<td></td>
</tr>
<tr>
<td>S₃</td>
<td>140</td>
<td>170</td>
<td>200</td>
<td></td>
</tr>
</tbody>
</table>

Maximum under each decision are (220, 190, 200). The maximum of these three maximums is 220. Consequently, according to the maximax criteria the decision to be adopted is A₁.

(b) **The minimax decision criterion** Minimax is just the opposite of maximax. Application of the minimax criteria requires a table of losses instead of gains. The losses are the costs to be incurred or the damages to be suffered for each of the alternative actions and states of nature. The minimax rule minimizes the maximum possible loss for each course of action. The term ‘minimax’ is an abbreviation of the phrase minimum of the maximum. Under each of the various actions, there is a maximum loss and the action that is associated with the minimum of the various maximum losses is the action to be taken according to the minimax criterion. Suppose the loss table is
It shows that the maximum losses incurred by the various decisions

\[
\begin{array}{ccc}
A_1 & A_2 & A_3 \\
18 & 14 & 10
\end{array}
\]

and the minimum among three maximums is 10 which is under action \( A_3 \). Thus, according to minimax criterion, the decision-maker should take action \( A_3 \).

(c) **The maximin decision criterion (criterion of pessimism)**  The maximin criterion of decision-making stands for choice between alternative courses of action assuming pessimistic view of nature. Taking each act in turn, we note the worst possible results in terms of pay-off and select the act which maximizes the minimum pay-off. Suppose the pay-off table is

<table>
<thead>
<tr>
<th>State of nature</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
<th>( S_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>-80</td>
<td>-60</td>
<td>-20</td>
<td></td>
</tr>
<tr>
<td>( A_2 )</td>
<td>-30</td>
<td>-10</td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>( A_3 )</td>
<td>30</td>
<td>15</td>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>

Minimum under each decision are respectively

\[
\begin{array}{ccc}
-80 & -60 & -20
\end{array}
\]

The action \( A_3 \) is to be taken according to this criterion because it is the maximum among minimums.

(d) **Laplace criterion**  As the decision-maker has no information about the probability of occurrence of various events, he makes a simple assumption that each probability is equally likely. The expected pay-off is worked out on the basis of these probabilities. The act having maximum expected pay-off is selected.

**Example 4.1:**

<table>
<thead>
<tr>
<th>Events</th>
<th>Act</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( A_1 )</td>
</tr>
<tr>
<td>( E_1 )</td>
<td>20</td>
</tr>
<tr>
<td>( E_2 )</td>
<td>25</td>
</tr>
<tr>
<td>( E_3 )</td>
<td>30</td>
</tr>
</tbody>
</table>
Solution: We associate equal probability for each event say, 1/3. Expected pay-offs are:

\[
A_1 \rightarrow \left(20 \times \frac{1}{3}\right) + \left(25 \times \frac{1}{3}\right) + \left(30 \times \frac{1}{3}\right) = \frac{75}{3} = 25
\]

\[
A_2 \rightarrow \left(12 \times \frac{1}{3}\right) + \left(15 \times \frac{1}{3}\right) + \left(20 \times \frac{1}{3}\right) = \frac{47}{3} = 15.67
\]

\[
A_3 \rightarrow \left(25 \times \frac{1}{3}\right) + \left(30 \times \frac{1}{3}\right) + \left(22 \times \frac{1}{3}\right) = \frac{77}{3} = 25.67
\]

Since \(A_3\) has the maximum expected pay-off, \(A_3\) is the optimal act.

(e) **Harwicz alpha criterion** This method is a combination of maximum criterion and maximax criterion. In this method, the decision-maker’s degree of optimism is represented by \(\alpha\), the coefficient of optimism. \(\alpha\) varies between 0 and 1. When \(\alpha = 0\), there is total pessimism and when \(\alpha = 1\), there is total optimism.

we find \(D_1, D_2, D_3\), etc. connected with all strategies where \(D_i = a M_i + (1 - a) m_i\) where \(M_i\) is the maximum pay-off of ‘i’ the strategy and \(m_i\) is the minimum pay-off of ‘i’ th strategy. The strategy with highest of \(D_i\), \(D_2\), ...... is chosen. The decision-maker will specify the value of \(\alpha\) depending upon his level of optimism.

**Example 4.2:**

<table>
<thead>
<tr>
<th>Events</th>
<th>Act</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1</td>
<td>20</td>
<td>12</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>E2</td>
<td>25</td>
<td>15</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>E3</td>
<td>30</td>
<td>20</td>
<td>22</td>
<td></td>
</tr>
</tbody>
</table>

Solution: Let \(\alpha = .6\)

for \(A_1\) max. pay-off = 30

Min. pay-off = 20

\[
D_i = (6 \times 30) + (1 - .6)20 = 26
\]

Similarly, \(D_2\)

\[
D_2 = .6 30 1 .6 12 22.8
\]

\[
D_3 = .6 30 1 .6 22 26.8
\]

Since \(D_3\) is max, select the act \(A_3\).

### 4.2.3 Decision-Making under Risk

In this situation, the decision-maker has to face several states of nature. But he has some knowledge or experience which will enable him to assign probability to the occurrence of each state of nature. The objective is to optimize the expected profit, or to minimize the opportunity loss.

For decision problems under risk, the most popular methods used are Expected Monetary Value (EMV) criterion, Expected Opportunity Loss (EOL) criterion or Expected Value of Perfect Information (EVPI).
(a) **EMV Criterion**  When probabilities can be assigned to the various states of nature, it is possible to calculate the statistical expectation of gain for each course of action.

The conditional value of each event in the pay-off table is multiplied by its probability and the product is summed up. The resulting number is the EMV for the act. The decision-maker then selects from the available alternative actions, the action that leads to the maximum expected gain (that is the action with highest EMV).

Consider the following example. Let the states of nature be $S_1$ and $S_2$ and the alternative strategies be $A_1$ and $A_2$. Let the pay-off table be as follows.

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>30</td>
<td>20</td>
</tr>
<tr>
<td>$S_2$</td>
<td>35</td>
<td>30</td>
</tr>
</tbody>
</table>

Let the probabilities for the states of nature $S_1$ and $S_2$ be 6 and 4 respectively.

Then,

- EMV for $A_1 = (30 \times .6) + (35 \times .4) = 18 + 14 = 32$
- EMV for $A_2 = (20 \times .6) + (30 \times .4) = 12 + 12 = 24$

Since EMV for $A_1$ is greater, the decision-maker will choose the strategy $A_1$.

(b) **EOL Criterion**  The difference between the greater pay-off and the actual pay-off is known as opportunity loss. Under this criterion, the strategy which has minimum Expected Opportunity Loss (EOL) is chosen. The calculation of EOL is similar to that of EMV.

The following is an opportunity loss table. $A_1$ and $A_2$ are the strategies and $S_1$ and $S_2$ are the states of nature.

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2</td>
<td>–5</td>
</tr>
</tbody>
</table>

Let the probabilities for two states be .6 and .4.

- EOL for $A_1 = (0 \times .6) + (2 \times .4) = 0.8$
- EOL for $A_2 = (10 \times .6) + (–5 \times .4) = 6 – 2 = 4$

EOL for $A_1$ is least. Therefore, the strategy $A_1$ may be chosen.

(c) **EVPI Method**  The expected value of perfect information (EVPI) is the average (expected) return in the long run, if we have perfect information before a decision is to be made.

In order to calculate EVPI, we choose the best alternative with the probability of their state of nature. EVPI is the expected outcome with perfect information minus the outcome with max EMV.

\[ \text{EVPI} = \text{Expected value with perfect information} – \text{max. EMV} \]

Consider the following example.
**Example 4.3:** $A_1, A_2, A_3$ are the acts and $S_1, S_2, S_3$ are the states of nature. Also, known that $P(S_1) = .5$, $P(S_2) = .4$ and $P(S_3) = .1$

**Solution:** Pay-off table is as follows:

<table>
<thead>
<tr>
<th>State of nature</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>30</td>
<td>25</td>
<td>22</td>
</tr>
<tr>
<td>$S_2$</td>
<td>20</td>
<td>35</td>
<td>20</td>
</tr>
<tr>
<td>$S_3$</td>
<td>40</td>
<td>30</td>
<td>35</td>
</tr>
</tbody>
</table>

EMV for $A_1 = (5 \times 30) + (4 \times 20) + (1 \times 40) = 15 + 8 + 4 = 27$

EMV for $A_2 = (5 \times 25) + (4 \times 35) + (1 \times 30) = 12.5 + 14 + 3 = 29.5$

EMV for $A_3 = (5 \times 22) + (4 \times 20) + (1 \times 35) = 11 + 8 + 3.5 = 22.5$

The highest EMV is for the strategy $A_2$ and it is 29.5.

Now to find EVPI, work out the expected value for maximum pay-off under all states of nature.

<table>
<thead>
<tr>
<th>Max. profit of each state</th>
<th>Probability</th>
<th>Expected value ($ = \text{Prob.} \times \text{Profit}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>30</td>
<td>.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>$S_2$</td>
<td>35</td>
<td>.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>14</td>
</tr>
<tr>
<td>$S_3$</td>
<td>40</td>
<td>.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

∴ Expected pay-off with perfect information = 33.

∴ Thus, the expected value of perfect information (EVPI) = Expected Value with Perfect Information – Maximum EMV = 33 – 29.5 = 3.5.

**Example 4.4:** You are given the following pay-offs of three acts $A_1, A_2$ and $A_3$, and the states of nature $S_1, S_2$ and $S_3$.

<table>
<thead>
<tr>
<th>States of nature</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>25</td>
<td>–10</td>
<td>–125</td>
</tr>
<tr>
<td>$S_2$</td>
<td>400</td>
<td>440</td>
<td>400</td>
</tr>
<tr>
<td>$S_3$</td>
<td>650</td>
<td>740</td>
<td>750</td>
</tr>
</tbody>
</table>

The probabilities of the states of nature are respectively, .1, .7 and .2. Calculate and tabulate the EMV and conclude which of the acts can be chosen as the best.

**Solution:**

<table>
<thead>
<tr>
<th>Act $A_1$</th>
<th>Prob. $\times$ Pay off</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1 $\times$ 25 = 25</td>
</tr>
<tr>
<td></td>
<td>0.7 $\times$ 400 = 280</td>
</tr>
<tr>
<td></td>
<td>0.2 $\times$ 650 = 130</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Act $A_2$</th>
<th>Prob. $\times$ Pay off</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1 $\times$ –10 = –1</td>
</tr>
<tr>
<td></td>
<td>0.7 $\times$ 440 = 308</td>
</tr>
<tr>
<td></td>
<td>0.2 $\times$ 740 = 148</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Act $A_3$</th>
<th>Prob. $\times$ Pay off</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1 $\times$ –125 = –12.5</td>
</tr>
<tr>
<td></td>
<td>0.7 $\times$ 400 = 280</td>
</tr>
<tr>
<td></td>
<td>0.2 $\times$ 750 = 150</td>
</tr>
</tbody>
</table>

∴ EMV for $A_1 = 412.5$, EMV for $A_2 = 455$, EMV for $A_3 = 417.5$

Since EMV is maximum for $A_2$ choose the Act $A_2$. 
Example 4.5: A management is faced with the problem of choosing one of the products for manufacturing. The probability matrix after market research for the two products was as follows.

<table>
<thead>
<tr>
<th>State of nature</th>
<th>Act</th>
<th>Good</th>
<th>Fair</th>
<th>Poor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product A</td>
<td>0.75</td>
<td>0.15</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>Product B</td>
<td>0.60</td>
<td>0.30</td>
<td>0.10</td>
<td></td>
</tr>
</tbody>
</table>

The profit that the management can make for different levels of market acceptability of the products are as follows.

<table>
<thead>
<tr>
<th>State of nature</th>
<th>Profit (in Rs) if market is</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acts</td>
<td>Good</td>
</tr>
<tr>
<td>Product A</td>
<td>35000</td>
</tr>
<tr>
<td>Product B</td>
<td>50000</td>
</tr>
</tbody>
</table>

Calculate the expected value of the choice of alternatives and advise the management.

Solution: Let us put this information in a pay-off matrix with probabilities associated with the states of nature.

<table>
<thead>
<tr>
<th>State of nature</th>
<th>Product A Profit × Probability</th>
<th>Product B Profit × Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good</td>
<td>35000 × 0.75 = 26250</td>
<td>50000 × 0.60 = 30000</td>
</tr>
<tr>
<td>Fair</td>
<td>1500 × 0.15 = 2250</td>
<td>20000 × 0.30 = 6000</td>
</tr>
<tr>
<td>Poor</td>
<td>5000 × 0.10 = 500</td>
<td>−3000 × 0.10 = −300</td>
</tr>
<tr>
<td>EMV</td>
<td>29000</td>
<td>35700</td>
</tr>
</tbody>
</table>

Since the expected pay-off (EMV) of product B is greater, product B should be preferred by the management.

Preparation of pay-off table

Example 4.6: An ink manufacturer produces a certain type of ink at a total average cost of Rs 3 per bottle and sells at a price of Rs 5 per bottle. The ink is produced over the weekend and is sold during the following week. According to past experience, the weekly demand has never been less than 78 or greater than 80 bottles in his place.

You are required to formulate the pay-off table.

Solution: The different states of nature are the demand for 78 units, 79 units or 80 units. Call them $S_1$, $S_2$, $S_3$.

The alternative courses of action are selling 78 units, 79 units or 80 units. Call them $A_1$, $A_2$, $A_3$.

Selling price of ink = Rs 5- per bottle
Cost price = Rs 3- per bottle
Calculation of pay-offs (Pay-off stands for the gain)
Sale quantity × price – Production quantity × Cost
\[ A_1S_1 = 78 \times 5 - 78 \times 3 = 390 - 234 = 156 \]
\[ A_2S_1 = 78 \times 5 - 79 \times 3 = 390 - 237 = 153 \]
\[ A_3S_1 = 78 \times 5 - 80 \times 3 = 390 - 240 = 150 \]
\[ A_1S_2 = 78 \times 5 - 78 \times 3 = 390 - 234 = 156 \]
\[ A_2S_2 = 79 \times 5 - 79 \times 3 = 395 - 237 = 158 \]
\[ A_3S_2 = 78 \times 5 - 80 \times 3 = 395 - 240 = 155 \]
\[ A_1S_3 = 78 \times 5 - 78 \times 3 = 390 - 234 = 156 \]
\[ A_2S_3 = 79 \times 5 - 79 \times 3 = 395 - 237 = 158 \]
\[ A_3S_3 = 80 \times 5 - 80 \times 3 = 400 - 240 = 160 \]

(Explanation: \( A_1S_1 \) means selling quantity is 78 and manufacturing quantity is 78. \( A_2S_1 \) means sales 78, production 79, and so on.)

Pay-off Table

<table>
<thead>
<tr>
<th>State of nature (events)</th>
<th>Act (strategy)</th>
<th>A_1</th>
<th>A_2</th>
<th>A_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>S_1</td>
<td></td>
<td>156</td>
<td>153</td>
<td>150</td>
</tr>
<tr>
<td>S_2</td>
<td></td>
<td>156</td>
<td>158</td>
<td>155</td>
</tr>
<tr>
<td>S_3</td>
<td></td>
<td>156</td>
<td>158</td>
<td>160</td>
</tr>
</tbody>
</table>

Note: We shall show the states of nature in rows and acts in columns.

Preparation of loss table

**Example 4.7:** An ink manufacturer produces a certain type of ink at a total average cost of Rs 3 per bottle and sells at a price of Rs 5 per bottle. The ink is produced over the weekend and is sold during the following week. According to past experience, the weekly demand has never been less than 78 or greater than 80 bottles in his place.

You are required to formulate the loss table.

**Solution:** Calculation of regret (opportunity loss)

\[ A_1S_1 = 0 \text{ (since production and sales are of equal quantity say 78)} \]
\[ A_2S_1 = 1 \times 3 = 3 \text{ (since one unit of production is in excess whose cost = Rs 3)} \]
\[ A_3S_1 = 2 \times 3 = 6 \text{ (since 2 units of production are in excess whose unit cost is @ Rs 3)} \]
\[ A_1S_2 = 1 \times 2 = 2 \text{ (since the demand of one unit is more than produced, the profit for one unit is Rs 2)} \]

Similarly, \( A_2S_2 = 0 \) (since units of production = units of demand)
\[ A_3S_2 = 2 \times 1 = 2 \quad \text{and} \quad A_3S_3 = 0 \]

Opportunity loss table

<table>
<thead>
<tr>
<th>State of nature (events)</th>
<th>Action</th>
<th>A_1</th>
<th>A_2</th>
<th>A_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>S_1</td>
<td></td>
<td>0</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>S_2</td>
<td></td>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>S_3</td>
<td></td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
From the pay-off table opportunity loss table can also be prepared.

**Method:** Let every row of the pay-off table represent a state of nature and every column represent a course of action. Then from each row select the highest pay-off and subtract all pay-offs of that row from it. They are the opportunity losses.

See the following examples.

**Example 4.8:** The following is a pay-off table. From it, form a regret (opportunity loss) table.

<table>
<thead>
<tr>
<th>States of nature</th>
<th>A₁</th>
<th>A₂</th>
<th>A₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>E₁</td>
<td>156</td>
<td>153</td>
<td>150</td>
</tr>
<tr>
<td>E₂</td>
<td>156</td>
<td>158</td>
<td>155</td>
</tr>
<tr>
<td>E₃</td>
<td>156</td>
<td>158</td>
<td>160</td>
</tr>
</tbody>
</table>

**Solution:**

<table>
<thead>
<tr>
<th>State of nature</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A₁</td>
</tr>
<tr>
<td>E₁</td>
<td>156 – 156 = 0</td>
</tr>
<tr>
<td>E₂</td>
<td>158 – 156 = 2</td>
</tr>
<tr>
<td>E₃</td>
<td>160 – 156 = 4</td>
</tr>
</tbody>
</table>

**Example 4.9:** The following is a pay-off table.

<table>
<thead>
<tr>
<th>Event (State of nature)</th>
<th>E₁</th>
<th>E₂</th>
<th>E₃</th>
<th>E₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
<td>50</td>
<td>300</td>
<td>–150</td>
<td>50</td>
</tr>
<tr>
<td>A₂</td>
<td>400</td>
<td>0</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>A₃</td>
<td>–50</td>
<td>200</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>A₄</td>
<td>0</td>
<td>300</td>
<td>300</td>
<td>0</td>
</tr>
</tbody>
</table>

Suppose that the probabilities of the events in this table are P(E₁) = 0.15; P(E₂) = 0.45; P(E₃) = 0.25; P(E₄) = 0.15.

Calculate the expected pay-off. Prepare the opportunity loss table (Regret table) and calculate the expected loss of each action.

**Solution:**

Hint: Rewrite the question with E₁, E₂, E₃, E₄ as rows and A₁, A₂, A₃, A₄ as columns.
Calculation of Expected Pay-off (EMV)

<table>
<thead>
<tr>
<th></th>
<th>A1</th>
<th></th>
<th>A2</th>
<th></th>
<th>A3</th>
<th></th>
<th>A4</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>50 × .15</td>
<td>= 7.5</td>
<td>400 × .15</td>
<td>= 60</td>
<td>50 × .15</td>
<td>= 7.5</td>
<td>0 × .15</td>
<td>= 0</td>
<td></td>
</tr>
<tr>
<td>300 × .45</td>
<td>= 135</td>
<td>0 × .45</td>
<td>= 0</td>
<td>200 × .45</td>
<td>= 90</td>
<td>300 × .45</td>
<td>= 135</td>
<td></td>
</tr>
<tr>
<td>−150 × .25</td>
<td>= −37.5</td>
<td>100 × .25</td>
<td>= 25</td>
<td>0 × .25</td>
<td>= 0</td>
<td>300 × .25</td>
<td>= 75</td>
<td></td>
</tr>
<tr>
<td>50 × .15</td>
<td>= 7.5</td>
<td>0 × .15</td>
<td>= 0</td>
<td>100 × .15</td>
<td>= 15</td>
<td>0 × .15</td>
<td>= 0</td>
<td></td>
</tr>
<tr>
<td>EMV = 112.5</td>
<td></td>
<td>EMV = 85</td>
<td></td>
<td>EMV = 112.5</td>
<td></td>
<td>EMV = 210</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Opportunity Loss Table

<table>
<thead>
<tr>
<th></th>
<th>A1</th>
<th></th>
<th>A2</th>
<th></th>
<th>A3</th>
<th></th>
<th>A4</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>E1</td>
<td>400−50  = 350</td>
<td>400−400 = 0</td>
<td>400+50 = 450</td>
<td>400−0   = 400</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E2</td>
<td>300−300 = 0</td>
<td>300−0   = 300</td>
<td>300−200 = 100</td>
<td>300−300 = 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E3</td>
<td>300+150 = 450</td>
<td>300−100 = 200</td>
<td>300−0   = 300</td>
<td>300−300 = 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E4</td>
<td>100−50 = 50</td>
<td>100−0   = 100</td>
<td>100−100 = 0</td>
<td>100−0   = 100</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Calculation Expected Loss (EOL)

<table>
<thead>
<tr>
<th></th>
<th>A1</th>
<th></th>
<th>A2</th>
<th></th>
<th>A3</th>
<th></th>
<th>A4</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>350 × .15</td>
<td>= 52.5</td>
<td>0 × .15  = 0</td>
<td>450 × .15</td>
<td>= 67.5</td>
<td>400 × .15</td>
<td>= 60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 × .45  = 0</td>
<td>300 × .45 = 135</td>
<td>100 × .45</td>
<td>= 45</td>
<td>0 × .45  = 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>450 × .25</td>
<td>= 112.5</td>
<td>200 × .25 = 50</td>
<td>300 × .25</td>
<td>= 75</td>
<td>0 × .25  = 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50 × .15</td>
<td>= 7.5</td>
<td>100 × .15 = 15</td>
<td>0 × .15  = 0</td>
<td>100 × .15= 15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EOL = 172.5</td>
<td></td>
<td>EOL = 200</td>
<td></td>
<td>EOL = 187.5</td>
<td></td>
<td>EOL = 75</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 4.10: A newspaper boy has the following probability of selling a magazine:

<table>
<thead>
<tr>
<th>No. of copies sold</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.10</td>
</tr>
<tr>
<td>11</td>
<td>0.15</td>
</tr>
<tr>
<td>12</td>
<td>0.20</td>
</tr>
<tr>
<td>13</td>
<td>0.25</td>
</tr>
<tr>
<td>14</td>
<td>0.30</td>
</tr>
</tbody>
</table>

The cost of a copy is 30 paise and sale price is 50 paise. He cannot return unsold copies. How many copies should he order?

Solution: We can apply either EMV criterion or EOL criterion. Let us apply EMV criterion for which we have to calculate the pay-off. Number of copies ordered are the different courses of action. The copies ordered may be 10, 11, 12, 13, 14. Denote them by A1, A2, A3, A4, A5.

Similarly, number of copies demanded may be 10, 11, 12, 13 or 14. These demands may be D1, D2, D3, D4, D5. These are events. The pay-off values are calculated as follows:

Selling price of each item = 50 paise and cost of a copy = 30 paise

\[ A_1D_1 = (10 \times 50) - (10 \times 30) = 200 \]
\[ A_2D_1 = (10 \times 50) - (11 \times 30) = 170 \]
\[ A_3D_1 = (10 \times 50) - (12 \times 30) = 140 \]
\[ A_4D_1 = (10 \times 50) - (13 \times 30) = 110 \]

and so on.
The 25 pay-off values can thus be calculated. They are shown as follows:

<table>
<thead>
<tr>
<th></th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
<th>( A_4 )</th>
<th>( A_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_1 )</td>
<td>200</td>
<td>170</td>
<td>140</td>
<td>110</td>
<td>80</td>
</tr>
<tr>
<td>( D_2 )</td>
<td>200</td>
<td>220</td>
<td>190</td>
<td>160</td>
<td>130</td>
</tr>
<tr>
<td>( D_3 )</td>
<td>200</td>
<td>220</td>
<td>240</td>
<td>210</td>
<td>180</td>
</tr>
<tr>
<td>( D_4 )</td>
<td>200</td>
<td>220</td>
<td>240</td>
<td>260</td>
<td>230</td>
</tr>
<tr>
<td>( D_5 )</td>
<td>200</td>
<td>220</td>
<td>240</td>
<td>260</td>
<td>280</td>
</tr>
</tbody>
</table>

The given probabilities are .10, .15, .20, .25, .30.

Calculation of EMV for all the acts.

\[
\begin{align*}
\text{A}_1 & : \quad \text{Pay off} \times \text{Prob.} \\
.10 \times 200 = 20 & \quad .10 \times 170 = 17 & \quad .10 \times 140 = 20 \\
.15 \times 200 = 30 & \quad .15 \times 220 = 33 & \quad .15 \times 190 = 28.5 \\
.20 \times 200 = 40 & \quad .20 \times 220 = 44 & \quad .20 \times 240 = 48 \\
.25 \times 200 = 50 & \quad .25 \times 220 = 55 & \quad .25 \times 240 = 60 \\
.30 \times 200 = 60 & \quad .30 \times 220 = 66 & \quad .30 \times 240 = 72 \\
200 & \quad 215 & \quad 222.5 \\
\end{align*}
\]

\[
\begin{align*}
\text{A}_4 & : \quad \text{Pay off} \times \text{Prob.} \\
.10 \times 110 = 20 & \quad .10 \times 80 = 8 \\
.15 \times 160 = 30 & \quad .15 \times 130 = 19.5 \\
.20 \times 210 = 40 & \quad .20 \times 180 = 36 \\
.25 \times 260 = 50 & \quad .25 \times 230 = 55 \\
.30 \times 260 = 50 & \quad .30 \times 280 = 84 \\
220 & \quad 205 \\
\end{align*}
\]

\[
\begin{align*}
\text{A}_5 & : \quad \text{Pay off} \times \text{Prob.} \\
.10 \times 110 = 20 & \quad .10 \times 80 = 8 \\
.15 \times 160 = 30 & \quad .15 \times 130 = 19.5 \\
.20 \times 210 = 40 & \quad .20 \times 180 = 36 \\
.25 \times 260 = 50 & \quad .25 \times 230 = 55 \\
.30 \times 260 = 50 & \quad .30 \times 280 = 84 \\
220 & \quad 205 \\
\end{align*}
\]

\[
\begin{align*}
\text{EMV for the acts } A_1 & : 200, \quad A_2 : 215, \quad A_3 : 222.5, \quad A_4 : 220, \quad A_5 : 205.
\end{align*}
\]

\[
\begin{align*}
\text{EMV for } A_3 \text{ is greater and therefore } A_3 \text{ is optimal act.}
\end{align*}
\]

\[
\begin{align*}
\text{No. of copies to be ordered } = 12.
\end{align*}
\]

**Example 4.11:** A grocery store with a bakery department is faced with the problem of how many cakes to buy in order to meet the day’s demand. The grocer prefers not to sell day-old goods in competition with fresh products. Leftover cakes are, therefore, a complete loss. On the other hand, if a customer desires a cake and all of them have been sold, the disappointed customer will buy elsewhere and the sales will be lost. The grocer has therefore collected information on the past sales in a selected 100-day period as shown in the following table:

<table>
<thead>
<tr>
<th>Sales per day</th>
<th>No. of days</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>10</td>
<td>0.10</td>
</tr>
<tr>
<td>26</td>
<td>30</td>
<td>0.30</td>
</tr>
<tr>
<td>27</td>
<td>50</td>
<td>0.50</td>
</tr>
<tr>
<td>28</td>
<td>10</td>
<td>0.10</td>
</tr>
<tr>
<td><strong>100</strong></td>
<td><strong>1.00</strong></td>
<td></td>
</tr>
</tbody>
</table>

Construct the pay-off table and the opportunity loss table. What is the optimal number of cakes that should be bought each day? Apply both EMV and EOL criteria.
Also find (and interpret) EVPI (Expected Value of Perfect Information). A cake costs Rs 0.80 and sells for Re 1.

**Solution:** Let \( A_1, A_2, A_3, \) and \( A_4 \) for strategies and \( S_1, S_2, S_3, S_4 \) stands for states of nature.

Then \( A_1, A_2, A_3, A_4 \) respectively stand for stocking 25, 26, 27, 28 cakes. \( S_1, S_2, S_3, S_4 \) respectively stand for demands for 25, 26, 27, 28 cakes.

Conditional pay-off values can be obtained as explained earlier. The pay-off values thus obtained are as follows.

**Pay-off table**

<table>
<thead>
<tr>
<th>State of nature (Demand)</th>
<th>Alternative strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_1(25) )</td>
<td>( A_1(25) ) 5.00</td>
</tr>
<tr>
<td>( S_2(26) )</td>
<td>( A_3(27) ) 3.40</td>
</tr>
<tr>
<td>( S_3(27) )</td>
<td>( A_1(25) ) 5.00</td>
</tr>
<tr>
<td>( S_4(28) )</td>
<td>( A_3(27) ) 5.40</td>
</tr>
</tbody>
</table>

Probabilities are .10, .30, .50, .10

\[ \text{EMV for Act } A_1 = (5 \times 0.10) + (5 \times 0.30) + (5 \times 0.50) + (5 \times 0.10) = .5 \]

Similarly, EMV values \( A_2, A_3, A_4 \) are 5.10, 4.9, 4.2 respectively.

The maximum EMV is seen strategy \( A_2 \). Thus, according to the EMV decision criterion, the store would stock 26 cakes (\( A_2 \)).

**Regret (Opportunity loss) table**

<table>
<thead>
<tr>
<th>State of nature (Demand)</th>
<th>Alternative strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_1(25) )</td>
<td>( A_1(25) ) 0</td>
</tr>
<tr>
<td>( S_2(26) )</td>
<td>( A_3(27) ) 1.6</td>
</tr>
<tr>
<td>( S_3(27) )</td>
<td>( A_4(28) ) 2.4</td>
</tr>
<tr>
<td>( S_4(28) )</td>
<td>( A_1(25) ) 0.2</td>
</tr>
</tbody>
</table>

Probabilities are 0.10, 0.30, 0.50, 0.10

Expected opportunity loss for \( A_1 = (0 \times 0.10) + (0.30 \times 0.30) + (2 \times 0.50) + (0 \times 0.10) = .32 \)

Similarly for \( A_2, A_3, A_4 \) expected opportunity losses are 0.22, 0.42 and 1.12 respectively.

EOL is least for \( A_2 \). \( A_2 \) is the optimal Act.

Hence, 26 is the optimal number to buy.

To find EVPI, select the highest pay-off in each row and find the expected value. Then we have

<table>
<thead>
<tr>
<th>Pay-off</th>
<th>Prob.</th>
<th>Payoff × Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.00</td>
<td>0.10</td>
<td>0.50</td>
</tr>
<tr>
<td>5.20</td>
<td>0.30</td>
<td>1.56</td>
</tr>
<tr>
<td>5.40</td>
<td>0.50</td>
<td>2.70</td>
</tr>
<tr>
<td>5.60</td>
<td>0.10</td>
<td>0.56</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td><strong>5.32</strong></td>
</tr>
</tbody>
</table>
.: Expected value with perfect information = 5.32

Max. EMV is for Act 𝑀₂ which is equal to 5.10.

Expected value of perfect information (EVPI) = Expected Value with Perfect Information – Highest EMV (EMV of 𝑀₂) = 5.32 – 5.10 = .32

Example 4.12: A factory produces 3 varieties of fountain pens. The fixed and variable costs are as follows:

<table>
<thead>
<tr>
<th>Fixed Cost</th>
<th>Variable Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 1 Rs 2,00,000</td>
<td>Rs 10</td>
</tr>
<tr>
<td>Type 2 Rs 3,20,000</td>
<td>Rs 8</td>
</tr>
<tr>
<td>Type 3 Rs 6,00,000</td>
<td>Rs 6</td>
</tr>
</tbody>
</table>

The likely demands under three situations are given as follows:

<table>
<thead>
<tr>
<th>Demand</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poor</td>
<td>25,000</td>
</tr>
<tr>
<td>Moderate</td>
<td>1,00,000</td>
</tr>
<tr>
<td>High</td>
<td>1,50,000</td>
</tr>
</tbody>
</table>

If the price of each type is Rs 20, prepare the pay-off table after showing necessary calculations.

**Solution:** Let 𝑇₁, 𝑇₂, 𝑇₃ stand for Type 1, Type 2, and Type 3 and 𝐷₁, 𝐷₂, and 𝐷₃ for poor demand, moderate demand and high demand respectively.

The pay-off (in thousands) = Sales revenue from the estimated demand – Total variable cost – Fixed cost

\[
\begin{align*}
T₁D₁ & = (20 \times 25) – (10 \times 25) – 200 = 500 – 250 – 200 = +50 \\
T₂D₁ & = (20 \times 25) – (8 \times 25) – 320 = 500 – 200 – 320 = –20 \\
T₃D₁ & = (20 \times 25) – (6 \times 25) – 600 = 500 – 150 – 600 = –250 \\
T₁D₂ & = (20 \times 100) – (10 \times 100) – 200 = 2000 – 1000 – 200 = +800 \\
T₂D₂ & = (20 \times 100) – (8 \times 100) – 320 = 2000 – 800 – 320 = +880 \\
T₃D₂ & = (20 \times 100) – (6 \times 100) – 600 = 2000 – 600 – 600 = +800 \\
T₁D₃ & = (20 \times 150) – (10 \times 150) – 200 = 3000 – 1500 – 200 = +1300 \\
T₂D₃ & = (20 \times 150) – (8 \times 150) – 320 = 3000 – 1200 – 320 = +1480 \\
T₃D₃ & = (20 \times 150) – (6 \times 150) – 600 = 3000 – 900 – 600 = +1500
\end{align*}
\]

The pay-off table (in ‘000s)

<table>
<thead>
<tr>
<th></th>
<th>𝑇₁</th>
<th>𝑇₂</th>
<th>𝑇₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>𝐷₁</td>
<td>50</td>
<td>–20</td>
<td>–250</td>
</tr>
<tr>
<td>𝐷₂</td>
<td>800</td>
<td>880</td>
<td>800</td>
</tr>
<tr>
<td>𝐷₃</td>
<td>1300</td>
<td>1480</td>
<td>1500</td>
</tr>
</tbody>
</table>

**Note:**

(i) Expected value of sales = Sum of the products of various values of sales and probabilities

(ii) Expected Monetary value of an act = Sum of the products of various values of the pay off of the act and probabilities

(iii) Expected value of cost = Sum of the products of the various values of the cost and probabilities

(iv) Expected value of loss of an act = Sum of the products of the losses of the acts and Probabilities
Example 4.13: RM (Mudland) Limited has recently installed new machinery but has not yet decided on the appropriate number of a certain spare part required for repairs.

Spare parts cost Rs 2,000 each but are only available if ordered now. If the plant failed and there was no spare part available, the cost to the business of mending the plant rises to Rs 15,000. The plant has an estimated life of ten years and the probability distribution of failure during this time, based on the experience with similar plant, is as follows.

<table>
<thead>
<tr>
<th>Number of failure over ten-year period</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>0.1</td>
</tr>
<tr>
<td>5 and over</td>
<td>Nil</td>
</tr>
</tbody>
</table>

(For the purpose of this question, ignore any discounting for time value of money)

You are required to calculate:

(a) The expected number of failures in the ten-year periods
(b) The optimal number of spares that should be purchased now: (Include in your workings a table to show the cost of alternative ordering policies and failures)
(c) The current cost of the ordering policy chosen;
(d) The value of perfect information of the number of failures in the ten-year life.

Solution:

(a) Expected number of failures

\[
= (0 \times 0.1) + (1 \times 0.4) + (2 \times 0.3) + (3 \times 0.1) + (4 \times 0.1) = 1.7
\]

(b) Pay-off values are calculated as follows:

<table>
<thead>
<tr>
<th>State of nature (Number of failure)</th>
<th>A_1 (0)</th>
<th>A_2 (1)</th>
<th>A_3 (2)</th>
<th>A_4 (3)</th>
<th>A_5 (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E_1(0)</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>E_2(1)</td>
<td>15</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>E_3(2)</td>
<td>30</td>
<td>17</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>E_4(3)</td>
<td>45</td>
<td>32</td>
<td>19</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>E_5(4)</td>
<td>60</td>
<td>47</td>
<td>34</td>
<td>21</td>
<td>8</td>
</tr>
</tbody>
</table>

Explanation for calculation of pay-off values: Calculation of

\[A_1 E_1 = (0 \times 2000) + (0 \times 15000) = 0\]
\[A_1 E_2 = (0 \times 2000) + (1 \times 15000) = 15000\]
\[A_1 E_3 = (2 \times 2000) + (0 \times 15000) = 4000\]
\[A_1 E_4 = (2 \times 2000) + (3 \times 15000) = 19000, etc.\]
EMV for each A1, A2, A3, A4 and A5 are 25.5, 14, 8.5, 7.5, 8 respectively selecting the act with least expected value, Optimal Act is A3.

∴ Purchase three parts

c) If three spare parts are purchased, the expected cost is a minimum and the cost current of the ordering policy chosen = 3 × 2,000 = Rs 6000.

d) If one had perfect information, in advance, of the number of failures that would occur one would purchase the corresponding number of spare parts and expected cost under perfect information (ECPI).

\[
\begin{array}{ccc}
0.1 & 0 & (0.4 & 2000) & (0.3 & 4000) \\
(0.1 & 6000) & (0.1 & 8000) & \end{array}
\]

Rs 3400

The expected value of perfect information = Optimal EMV – ECPI = Rs 7500 – Rs 3,400 = 4,100.

Example 4.14: A food products company is counterplanning the introduction of a revolutionary new product with new packing to replace the existing product at much higher price (S1) or a moderate change in the composition of the existing product with a new packaging at a small increase in price (S2) or a small change in the composition of the existing expect the word, ‘New with a negligible increase in the price (S3). The three possible states of nature of events are: (i) high increase in sales (N1). (ii) no change in sales (N2) and (iii) decrease in sales (N3). The marketing department of the company worked out the pay-offs in terms of yearly new profits for each of the strategies or these events (expected sales). This is represented in the following table.

<table>
<thead>
<tr>
<th>Strategies</th>
<th>State of nature</th>
<th>Pay-offs</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>N1</td>
<td>700</td>
</tr>
<tr>
<td></td>
<td>N2</td>
<td>300</td>
</tr>
<tr>
<td></td>
<td>N3</td>
<td>150</td>
</tr>
<tr>
<td>S2</td>
<td>500</td>
<td></td>
</tr>
<tr>
<td></td>
<td>450</td>
<td></td>
</tr>
<tr>
<td>S3</td>
<td>300</td>
<td></td>
</tr>
<tr>
<td></td>
<td>300</td>
<td></td>
</tr>
</tbody>
</table>

Which strategy should the executive concerned choose on the basis of:

(a) Maximum criterion (b) Maximax criterion (c) Minimax regret criterion (d) Laplace criterion?

Solution: Writing the pay-off table properly

<table>
<thead>
<tr>
<th>State of nature</th>
<th>Strategies</th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
</tr>
</thead>
<tbody>
<tr>
<td>N1</td>
<td>700</td>
<td>500</td>
<td>300</td>
<td></td>
</tr>
<tr>
<td>N2</td>
<td>300</td>
<td>450</td>
<td>300</td>
<td></td>
</tr>
<tr>
<td>N3</td>
<td>150</td>
<td>0</td>
<td>300</td>
<td></td>
</tr>
</tbody>
</table>

(a) Maximin criterion

Minimum pay offs

\[
\begin{array}{c}
S_1 \\
S_2 \\
S_3 \\
\end{array}
\]

150
0
300

Max of these Minima = 300

The executive should choose strategy \( S_3 \).
(b) Maximax criterion

Maximum pay-offs

<table>
<thead>
<tr>
<th></th>
<th>S₁</th>
<th>S₂</th>
<th>S₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>N₁</td>
<td>700</td>
<td>500</td>
<td>300</td>
</tr>
<tr>
<td>N₂</td>
<td>450</td>
<td>450</td>
<td>300</td>
</tr>
<tr>
<td>N₃</td>
<td>300</td>
<td>300</td>
<td>300</td>
</tr>
</tbody>
</table>

Maximum of maxima = 700

∴ The executive can choose Act S₁.

(c) Minimax regret criterion

<table>
<thead>
<tr>
<th></th>
<th>S₁</th>
<th>S₂</th>
<th>S₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>N₁</td>
<td>700 – 700 = 0</td>
<td>700 – 500 = 200</td>
<td>700 – 300 = 400</td>
</tr>
<tr>
<td>N₂</td>
<td>450 – 300 = 150</td>
<td>450 – 450 = 0</td>
<td>450 – 300 = 150</td>
</tr>
<tr>
<td>N₃</td>
<td>300 – 150 = 150</td>
<td>300 – 0 = 300</td>
<td>300 – 300 = 0</td>
</tr>
</tbody>
</table>

Maximum Opportunity loss

- for S₁ = 150
- for S₂ = 300
- for S₃ = 400

The executive should choose strategy S₁ since it minimizes the maximum of the losses.

(d) Laplace criterion

Assigning equal probability (say 1/3) to each state of nature calculate the expected monetary value for each act.

\[
\text{Prob. } \times \text{ Pay-off:} \\
\left(\frac{1}{3} \times 700\right) + \left(\frac{1}{3} \times 300\right) + \left(\frac{1}{3} \times 150\right) = \frac{1150}{3} = 383.33 \\
\left(\frac{1}{3} \times 500\right) + \left(\frac{1}{3} \times 450\right) + \left(\frac{1}{3} \times 0\right) = \frac{950}{3} = 316.666 \\
\left(\frac{1}{3} \times 300\right) + \left(\frac{1}{3} \times 300\right) + \left(\frac{1}{3} \times 300\right) = \frac{900}{3} = 300
\]

Since the EMV is highest for strategy 1, the executive should select strategy S₁.

**Example 4.15:** The research department of consumer products division has recommended to the marketing department to launch a soap with three different perfumes. The marketing manager has to decide the type of perfume to launch under the following estimated pay-off for the various levels of sales.

<table>
<thead>
<tr>
<th>Estimated level of sales (units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Types of perfume</td>
</tr>
<tr>
<td>I</td>
</tr>
<tr>
<td>II</td>
</tr>
<tr>
<td>III</td>
</tr>
</tbody>
</table>

Examine which type can be chosen under maximax, minimax, maximin and Laplace criterion.
Solution: Rewriting the pay off table

<table>
<thead>
<tr>
<th>Levels of Sales</th>
<th>Types of perfume</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
</tr>
<tr>
<td>20000</td>
<td>250</td>
</tr>
<tr>
<td>10000</td>
<td>15</td>
</tr>
<tr>
<td>2000</td>
<td>10</td>
</tr>
</tbody>
</table>

(1) Maximax criterion
Maximum for Type I = 250
Maximum for Type II = 40
Maximum for Type III = 60
Maximum of maximum = 250
∴ Select type I perfume

(2) Minimax criterion
Loss table is

<table>
<thead>
<tr>
<th>Sales</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
</tr>
<tr>
<td>20,000</td>
<td>0</td>
</tr>
<tr>
<td>10,000</td>
<td>10</td>
</tr>
<tr>
<td>2,000</td>
<td>0</td>
</tr>
</tbody>
</table>

Maximum losses under I, II, III are respectively 10, 210, 190 (from the pay-off table)
Minimum of these is 10.
∴ Type I is preferred.

(3) Maximin criterion
Minimum pay-off under each type I, II, III are respectively 10, 5, 3 (from the pay-off table)
Maximum of these is 10.
∴ Type I is preferred.

(4) Laplace criterion
Let the probability for each levels of sales be taken as 1/3 each.
Expected pay-offs are:

Type I $\rightarrow \left(250 \times \frac{1}{3}\right) + \left(15 \times \frac{1}{3}\right) + \left(10 \times \frac{1}{3}\right) = \frac{275}{3} = 91.67$
Type II $\rightarrow \left(40 \times \frac{1}{3}\right) + \left(20 \times \frac{1}{3}\right) + \left(5 \times \frac{1}{3}\right) = \frac{65}{3} = 21.67$
Type III $\rightarrow \left(60 \times \frac{1}{3}\right) + \left(25 \times \frac{1}{3}\right) + \left(3 \times \frac{1}{3}\right) = \frac{88}{3} = 29.33$

Maximum expected pay-off is for Type I, so choose Type I perfume.
Example 4.16: A company has an opportunity to computerize its records department. However, the existing personnel have job security under union agreement. The cost of the three alternative programmes for the changeover depend upon the attitude of the union and is estimated as follows:

<table>
<thead>
<tr>
<th>Attitude of union</th>
<th>General retraining</th>
<th>Selective retraining</th>
<th>Hire new employees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Antagonist</td>
<td>940</td>
<td>920</td>
<td>900</td>
</tr>
<tr>
<td>Passive</td>
<td>810</td>
<td>800</td>
<td>820</td>
</tr>
<tr>
<td>Enthusiastic</td>
<td>700</td>
<td>710</td>
<td>860</td>
</tr>
</tbody>
</table>

(i) Construct the opportunity losses table.

Solution: Opportunity loss for row I is obtained by subtracting 900 (least element of row I) from all the elements of I row are

\[ 940 - 900 = 40, \quad 920 - 900 = 20, \quad 900 - 900 = 0 \]

Similarly, for row II the opportunity losses are

\[ 810 - 800 = 10, \quad 800 - 800 = 0 \text{ and } 820 - 800 = 20 \]

Similarly for row III, they are

\[ 700 - 700 = 0, \quad 710 - 700 = 10, \quad 860 - 700 = 160 \]

Thus, the opportunity loss table is

<table>
<thead>
<tr>
<th>Attitude of the nation</th>
<th>State of action</th>
<th>General</th>
<th>Selective</th>
<th>Hire new</th>
</tr>
</thead>
<tbody>
<tr>
<td>Antagonist</td>
<td></td>
<td>40</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>Passive</td>
<td></td>
<td>10</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>Enthusiastic</td>
<td></td>
<td>0</td>
<td>10</td>
<td>160</td>
</tr>
</tbody>
</table>

4.3 DECISION TREE ANALYSIS

A ‘Decision tree’ is one of the tools used for the diagramatic presentation of the sequential and multidimensional aspects of a particular decision problem for systematic analysis and evaluation. Here, the decision problem, the alternative courses of action, the states of nature and the likely outcomes of alternatives are diagramatically or graphically depicted as branches and sub-branches of a horizontal tree.

The decision tree consists of nodes and branches. The nodes are of two types, decision nodes and chance nodes. Courses of action (or strategies) originate from the decision nodes as the main branches. At the terminal of each main branch there are chance nodes. From these chance nodes, chance events emanate in the form of sub-branches. The respective pay-offs and probabilities associated with alternative courses and chances events are shown along the sub-branches. At the terminal of the sub-branches are shown the expected values of the outcome.
Here $A_1, A_2, A_3$ and $A_4$ are strategies $E_1, E_2, E_3$ are events $O_{11}, O_{12}, O_{21}, O_{22}, O_{31}, O_{32}$ are outcomes.

A decision tree is highly useful to a decision-maker in multistage situations which involve a series of decisions each dependent on the preceding one. Working backward, from the future to the present, we are able to eliminate unprofitable branches and determine optimum decisions. The decision tree analysis allows one to understand, simply by inspection, various assumptions and alternatives in a graphic form which is much more easier to understand than the abstract analytical form.

The advantages of the decision tree structure are that complex managerial problems and decisions of a chain-like nature can be systematically and explicitly defined and evaluated.

**Example 4.17:** A firm owner is seriously considering to drill a farm well. In the past, only 70% of wells drilled were successful at 200 feet of depth in the area. Moreover, on finding no water at 200 ft., some persons drilled it further up to 250 feet but only 20% struck water at 250 ft. The prevailing cost of drilling is Rs 50 per foot. The farm owner has estimated that in case he does not get his own wells, he will have to pay Rs 15,000 over the next 10 years, in present value (PV) term, to buy water from the neighbour. The following decisions can be optimal.

(i) Do not drill any well (ii) drill up to 200 ft and (iii) if no water is found at 200 ft. drill further up to 250 ft.

Draw an appropriate decision tree and determine the farm owner’s strategy under EMV approach.

**Solution:**

![Decision Tree for the Firm Owner](image)
Probabilities are .2, .8

EMV for drill up to 250 feet.
\[= (12500 \times .2) + (27500 \times .8) = 24500\]

EMV for do not drill = 25000 (from the tree)
EMV is smaller for the act drill upto 250 feet. So it is an optimal act.

At D₁ point
The decisions are drill upto 200 feet and do not drill. Events are same as those of D₂ point.
Probabilities are .7, .3

EMV for drill up to 200 feet
\[= (10000 \times .7) + (24500 \times .3) = 14350\]

EMV for do not drill = 15,000 from the tree.
The optimal decision is drill upto 200 feet (as the EMV is small).

Therefore, combining D₁ and D₂, the optimal strategy is to drill the well upto 200 feet and if no water is struck, then further drill it to 250 feet.

**Example 4.18:** A firm is planning to develop and market a new drug. The cost of extensive research to develop the drug has been estimated at Rs. 100000. The manager of the research programme has found that there is a 60% chance that the drug will be developed successfully. The market potential has been assessed as follows:

<table>
<thead>
<tr>
<th>Market condition</th>
<th>Prob.</th>
<th>Present value of profit (Rs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large Market potential</td>
<td>0.1</td>
<td>50,000</td>
</tr>
<tr>
<td>Moderate Market potential</td>
<td>0.6</td>
<td>25,000</td>
</tr>
<tr>
<td>Low Market potential</td>
<td>0.3</td>
<td>10,000</td>
</tr>
</tbody>
</table>

The present value figures do not include the cost of research. While the firm is considering this proposal, a second proposal almost similar comes up for consideration. The second one also requires an investment of Rs 1,00,000 but the present value of all profits is Rs 12,000. The return on investment in the second proposal is certain.

(i) Draw a decision tree indicating all events and choices of the firm.
(ii) What decision should the firm take regarding the investment of Rs 1,00,000?

**Solution:**
At point D₂
Decisions are (1) Enter market (2) Do not enter market.
(a) Enter market
   \[ \text{EMV} = \text{Expected PV} = (50000 \times .1) + (25000 \times .6) + (10000 \times .3) = 5000 + 15000 + 3000 = 23000 \]
(b) Do not enter market
   \[ \text{EMV} = \text{Expected PV} = 0 \times 1 = 0 \]
Decision: Enter the market since EMV is more.

At point D₁
Decisions are (1) Develop new drug (2) Accept proposal II
(a) Develop new drug
   \[ \text{EMV} = \text{Expected PV} = (23000 \times .6) + (0 \times .4) = 13800 + 0 = 13800 \]
(b) Accept Proposal II
   \[ \text{EMV} = \text{Expected PV} = 12000 \times 1 = 12000 \]
Using EMV criterion, the optimal decision at D₁ is to develop and market the new drug.

4.4 INTEGER PROGRAMMING AND DYNAMIC PROGRAMMING: CONCEPTS AND ADVANTAGES

A linear programming problem in which all or some of the decision variables are constrained to assume non-negative integer values is called an Integer Programming Problem. (IPP).

In a linear programming problem if all variables are required to take integral values then it is called the pure (all) integer programming problem (Pure IPP).

If only some of the variables in the optimal solution of an IPP are restricted to assume non-negative integer values while the remaining variables are free to take any non-negative values, then it is called a mixed integer programming problem (Mixed IPP).

Further, if all the variables in the optimal solution are allowed to take values 0 or 1, then the problem is called the 0–1 programming problem or standard discrete programming problem.

The general integer programming problem is given by Max \( Z = CX \).
Subject to the constraints
\[ A \ x \leq b \]
\[ x \geq 0 \] and some or all variables are integers
4.4.1 Importance of Integer Programming Problems

In IPP, all the decision variables were allowed to take any non-negative real values as it is quite possible and appropriate to have fractional values in many situations. There are several frequently occurring circumstances in business and industry that lead to planning models involving integer valued variables. For example, in production, manufacturing is frequently scheduled in terms of batches, lots or runs. In allocation of goods, a shipment must involve a discrete number of trucks and aircrafts. In such cases, the fractional values of variables like 13/3 may be meaningless in the context of the actual decision problem.

This is the main reason why integer programming is so important for marginal decisions.

4.4.2 Applications of Integer Programming

Integer programming is applied in business and industry. All assignment and transportation problems are integer programming problems, as in assignment and travelling salesmen problem, all the decision variables are either zero or one.

\[ x_{ij} = 0 \text{ or } 1 \]

Other examples are capital budgeting and production scheduling problems. In fact, any situation involving decisions of the type ‘either to do a job or not to do’ can be viewed as an IPP. In all such situations

\[ x_{ij} = 1 \text{ if the } j^{th} \text{ activity is performed,} \]
\[ 0 \text{ if the } j^{th} \text{ activity is not performed.} \]

In addition, allocation problems involving the allocation of men, and machines give rise to IPP, since such communities can be assigned only in integers and not in fractions.

Note: If the non-integer variable is rounded off, it violates the feasibility and also there is no guarantee that the rounded off solution will be optimal. Due to these difficulties, there is a need for developing a systematic and efficient procedure for obtaining the exact optimal integer solution to such problems.

Many decision-making problems involve a process that takes place in such a way that at each stage, the process is dependent on the strategy chosen. Such type of problems are called Dynamic Programming Problem (DPP). Thus, DPP is concerned with the theory of multistage decision process. Mathematically, a DPP is a decision making problem in \( n \) variables, the problem being subdivided into \( n \) sub-problems (segments), each sub-problem being a decision-making problem in one variable only. The solution to a DPP is achieved sequentially, starting from one (initial) stage to the next, till the final stage is reached.

4.4.3 Decision Tree and Bellman’s Principle of Optimality

A multistage decision system in which each decision and state variable can take only finite number of values can be represented graphically by a decision tree.
Circles represent nodes corresponding to stages and lines between circles denote arcs corresponding to decisions.

The dynamic programming technique deals with such situations by dividing the given problem into sub-problems or stages.

**Bellman’s principle of optimality** states that ‘An optimal policy (set of decisions) has the property that whatever the initial state and decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.’

The problem which does not satisfy the principle of optimality cannot be solved by the dynamic programming method.

### 4.4.4 Characteristics of DPP

The basic features which characterize the dynamic programming problem are as follows.

1. The problem can be divided into stages with a policy decision required at each stage.
2. Every stage consists of a number of states associated with it. The states are the different possible conditions in which the system may find itself at that stage of the problem.
3. Decision at each stage converts the current stage into the state associated with the next stage.
4. The state of the system at a stage is described by a set of variables called state variables.
5. When the current state is known, an optimal policy for the remaining stages is independent of the policy of the previous ones.
6. The solution procedure begins by finding the optimal policy for each state to the last stage.
7. A recursive relationship which identifies the optimal policy for each state with \( n \) stages remaining, given the optimal policy for each state with \((n-1)\) stages left.
8. Using recursive equation approach, each time the solution procedure moves backward stage by stage for obtaining the optimum policy of each state for the particular stage, till it attains the optimum policy beginning at the initial stage.

**Note:** A stage may be defined as the portion of the problem that possesses a set of mutually exclusive alternatives from which the best alternative is to be selected.
4.4.5 Dynamic Programming Algorithm

The solution of a multistage problem by dynamic programming involves the following steps.

**Step 1:** Identify the decision variables and specify the objective function to be optimized under certain limitations, if any.

**Step 2:** Decompose the given problem into a number of smaller sub-problems. Identify the state variable at each stage.

**Step 3:** Write down the general recursive relationship for computing the optimal policy. Decide whether forward or backward method is to be followed to solve the problem.

**Step 4:** Construct appropriate stages to show the required values of the return function at each stage.

**Step 5:** Determine the overall optimal policy or decisions and its value at each stage. There may be more than such optimal policy.

4.4.6 Advantages of DPP

1. Generally the solution of a recursive equation involves two types of computations depending upon whether the system is continuous or discrete. In the first case, the optimal decision at each stage is obtained by using the usual classical method of optimization. In the second case, a tabular computational scheme is followed.

2. The D.P.P is solved by using the recursive equation, starting from the first to the last stage i.e. obtaining the sequence \( f_1, f_2, f_3, \ldots, f_n \) of the optimal solution. This computation is called the forward computational procedure. If the recursive equations are formulated to obtain a sequence \( f_n, f_{n-1}, \ldots, f_1 \) then the computation is known as the backward computational procedure.

**Example 4.19:** Use the principle of optimality to find the maximum value of 
\[
Z = b_1x_1 + b_2x_2 + \ldots + b_nx_n
\]
when 
\[
x_1 + x_2 + x_3 + \ldots + x_n = C
\]
\[
x_1, x_2, \ldots, x_n \geq 0
\]

**Solution:** The problem can be considered to divide the positive quantity \( C \) in \( n \) parts \( x_1, x_2, \ldots, x_n \) so that \( b_1x_1 + b_2x_2 + \ldots + b_nx_n \) is maximum.

Let \( f_n(C) = b_1x_1 + b_2x_2 + \ldots + b_nx_n \).

**Recursive equation**

If \( Z_i \) be the \( i^{th} \) part \((i = 1, 2 \ldots n)\) of the quantity then the recursive equation of the problem are

\[
f_i(x_i) = \max \{b_iZ_i = b_ix_i \}
\]
\[
Z_i = x_i
\]

and

\[
f(x_i) = \max \{b_iZ_i + f_{i-1}(x_i - Z_i)\}
\]

\[0 \leq Z_i \leq x_i[\text{where } i = 1, 2 \ldots n]\]
Solution of Recursive Equations

For one stage problem \( i = 1 \)

\[
f_i(x_i) = b_i x_i
\]

This gives, \( f_1(C) = b_1 C \) which is trivially true.

For two stage problem, where \( i = 1, 2 \)

\[
f_2(x_2) = \max \left( b_2 Z_2 + f_1(x_2 - Z_2) \right)
\]

\[
0 \leq Z_2 \leq x_2
\]

\[
f_2(C) = \max \left( b_2 Z_2 + f_1(C - Z) \right)
\]

\[
0 \leq Z_2 \leq C
\]

for \( x_2 = C \), \( Z_2 = Z \)

\[
= \max b_2 Z + b_1 (C - Z)
\]

\[
0 \leq Z \leq C
\]

\[
= \max (b_2 - b_1) Z + b_1 C
\]

\[
0 \leq Z \leq C
\]

If \( b_2 - b_1 \) is positive then this is maximum for \( Z = C \), otherwise, it will be minimum.

Thus \( f_2(C) = b_2 C \)

Similarly for three stage problem ( \( i = 1, 2, 3 \) )

\[
f_3(x_3) = \max \left( [(b_3 Z_3 + f_2(x_3 - Z_3)] \right)
\]

\[
0 \leq Z_3 \leq x_3
\]

\[
f_3(C) = \max \left( [(b_3 Z + f_2(C - Z)] \right)
\]

\[
0 \leq Z_3 \leq C
\]

\[
= \max [b_3 Z + b_2 (C - Z)]
\]

\[
0 \leq Z \leq C
\]

\[
= \max (b_3 - b_2) C + b_2 C
\]

\[
0 \leq Z \leq C
\]

Again if \( b_3 - b_2 \) is positive, then it gives maximum value for \( Z = C \) otherwise, it gives the minimum value.

Thus \( f_3(C) = b_3 C \)

From the results of the three stages 1, 2, 3 it can be easily shown by induction method that

\[
f_n(C) = b_n C
\]

Hence, the optimal policy will be

\( (0, 0, 0 \ldots x_n = C) \) with \( f_1(C) = b_n C \)
Example 4.20: Use dynamic programming to show that

\[ Z = p_1 \log p_1 + p_2 \log p_2 + \ldots + p_n \log p_n \]

Subject to the constraints \( p_1 + p_2 + \ldots + p_n = 1 \) and \( P_j \geq 0 \) \((j = 1, 2 \ldots n)\) is minimum, where \( p_1 = p_2 = \ldots = p_n = \frac{1}{n} \)

**Solution.** The problem here is to divide unity into \( n \) parts so as to minimize the quantity

\[ \Sigma p_j \log p_j \]

Let \( f_n(1) \) denote the minimum attainable sum of

\[ p_j \log p_j \quad (i = 1, 2, \ldots n) \]

For \( n = 1 \) (Stage 1)

\[ f_1(1) = \min \left( p_1 \log p_1 \right) = 1 \log 1 \]

\[ 0 < x \leq 1 \]

as unity is divided only into \( p_1 = 1 \) part

For \( n = 2 \), the unity is divided into two parts \( p_1 \) and \( p_2 \) such that \( p_1 + p_2 = 1 \)

If \( p_1 = x, p_2 = 1 - x \), then

\[ f_2(1) = \min \left( p_1 \log p_1 + p_2 \log p_2 \right) \]

\[ 0 \leq x \leq 1 \]

\[ = \min \left[ x \log x + (1 - x) \log (1 - x) \right] \]

\[ 0 \leq x \leq 1 \]

\[ = \min \left[ x \log x f_1(1 - x) \right] \]

\[ 0 \leq x \leq 1 \]

In general for an \( n \) stage problem the recursive equation is

\[ f_n(1) = \min \left( p_1 \log p_1 + p_2 \log p_2 + \ldots + p_n \log p_n \right) \]

\[ 0 \leq x \leq 1 \]

\[ = \min \left[ x \log x f_{n-1}(1 - x) \right] \]

\[ 0 \leq x \leq 1 \]

We solve this recursive equation

For \( n = 2 \), (Stage 2)

The function \( x \log x + (1 - x) \log (1 - x) \) attains its minimum value at

\[ x = \frac{1}{2} \]

satisfying the condition \( 0 < x \leq 1 \)

\[ f_2(1) = \frac{1}{2} \log \frac{1}{2} + \left( 1 - \frac{1}{2} \right) \log \left( 1 - \frac{1}{2} \right) = 2 \left( \frac{1}{2} \log \frac{1}{2} \right) \]

Similarly for stage 3, the minimum value of the recursive equation is obtained as

\[ f_3(1) = \min \left[ x \log x + f_2(1 - x) \right] \]

\[ 0 \leq x \leq 1 \]
NOTES

\[ = \min \left[ x \log x + 2 \left( \frac{1-x}{2} \right) \log \left( \frac{1-x}{2} \right) \right] \]
\[ 0 \leq x \leq 1 \]

Now, since the minimum value of

\[ x \log x + 2 \left( \frac{1-x}{2} \right) \log \left( \frac{1-x}{2} \right) \]

attained at \( x = \frac{1}{3} \) satisfying \( 0 < x \leq 1 \) we have

\[ f_3(1) = \frac{1}{3} \log \frac{1}{3} + 2 \left( \frac{1}{3} \log \frac{1}{3} \right) = \frac{3}{3} \log \frac{1}{3} \]

\[ \therefore \text{optimal policy is } P_1 = P_2 = P_3 = \frac{1}{3} \]

In general for \( n \) stage problem, we assume the optimal policy to be

\[ P_1 = P_2 = ... P_n = \frac{1}{n} \text{ and } f_n(1) = n \left[ \frac{1}{n} \log \frac{1}{n} \right] \]

This can be shown easily using mathematical induction.

For \( n = m + 1 \), the recursive equation is

\[ f_{m+1}(1) = \min \left[ x \log x + f_m(1-x) \right] \]
\[ 0 \leq x \leq 1 \]

\[ = \min \left[ x \log x + m \left( \frac{1-x}{m} \right) \log \left( \frac{1-x}{m} \right) \right] \]
\[ 0 \leq x \leq 1 \]

\[ = \frac{1}{m+1} \log \frac{1}{m+1} + m \left( \frac{1}{m+1} \log \frac{1}{m+1} \right) \]

\[ = m + 1 \left( \frac{1}{m+1} \log \frac{1}{m+1} \right) \]

Since minimum \( x \log x + \frac{1-x}{x} \log \frac{1-x}{m} \) is attained at \( x = \frac{1}{m+1} \).

Hence, the required optimum policy is \( \left( \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right) \) with \( f_n(1) = n \left( \frac{1}{n} \log \frac{1}{n} \right) \)
4.5 SUMMARY

In this unit, you have learned about decision-making and its applications. In any decision problem, the decision-maker is concerned with choosing the method which yields the best result from among the various alternative courses of action. The unit has explained that when the decision-maker faces multiple states of nature and has no means to arrive at probability values of the likelihood of occurrence of these states of nature, the problem is a decision problem under uncertainty. You have also learned that during decision-making under risk, the decision-maker has some knowledge which enables him to assign probability to the occurrence of each state of nature. You have also learned how to prepare a pay-off table. It has also described decision tree analysis, where you have learned that the decision problem, alternative courses of action, states of nature and the likely outcomes of alternatives are depicted as if they are branches and sub-branches of a horizontal tree. The unit has also explained the characteristics of integer programming and dynamic programming.

4.6 KEY TERMS

- **Tactical decisions**: Decisions which affect a business in the short run.
- **Strategic decisions**: Decisions which have far-reaching effects on the course of business.
- **Pay-off table**: It represents the economics of a problem, i.e., revenue and costs associated with any action with a particular outcome. It is an ordered statement of profit or costs resulting under the given situation.
- **Opportunity loss table**: It is the loss incurred because of failure to take the best possible action. Opportunity losses are calculated separately for each state of nature that might occur.
- **Decision-making under certainty**: In this case, the decision-maker knows with certainty the consequences of every alternative or decision choice. The decision-maker presumes that only one state of nature is relevant for his purpose.
- **Decision-making under uncertainty**: When the decision-maker faces multiple states of nature but he has no means to arrive at probability values to the likelihood of occurrence of these states of nature, the problem is a decision problem under uncertainty.
- **Decision tree**: A diagrammatic presentation of the sequential and multidimensional aspects of a particular decision problem for systematic analysis and evaluation.
- **Pure IPP**: In a linear programming problem if all variables are required to take integral values then it is called the pure (all) integer programming problem (Pure IPP).
- **Mixed IPP**: If only some variables in the optimal solution of an IPP are restricted to assume non-negative integer values while the remaining variables are free to take any non-negative values, then it is called a mixed integer programming problem (Mixed IPP).
4.7 ANSWERS TO ‘CHECK YOUR PROGRESS’

1. Tactical and strategic.
2. Events are the occurrences which affect the achievement of objectives. They are also called states of nature or outcomes.
4. Expected Monetary Value criterion and Expected Opportunity Loss criterion.
5. A decision tree is a diagrammatic presentation of the sequential and multidimensional aspects of a particular decision problem for systematic analysis and evaluation.
6. Complex managerial problems can be systematically and explicitly defined and evaluated.

4.8 QUESTIONS AND EXERCISES

Short-Answer Questions

1. What do you understand by ‘decision theory’?
2. What are EMV and EOL criteria?
3. What are pay-off tables and regret tables?

Long-Answer Questions

1. Describe some methods which are useful for decision-making under uncertainty.
2. Explain the terms (i) Expected Monetary Value (ii) Expected Value of Perfect Information.
3. Write notes on the Bayesian decision theory.
4. Write a note on decision tree.
5. Write a note on action space.
6. Explain (a) Maximax (b) Minimax (c) Maximin decision criteria.
7. From the following pay-off table, decide which is the optimal act.

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Events</th>
<th>A₁</th>
<th>A₂</th>
<th>A₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>S₁</td>
<td>20</td>
<td>40</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>S₂</td>
<td>15</td>
<td>−10</td>
<td>−15</td>
<td></td>
</tr>
<tr>
<td>S₃</td>
<td>35</td>
<td>25</td>
<td>−20</td>
<td></td>
</tr>
<tr>
<td>P(S₁)</td>
<td>.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P(S₂)</td>
<td>.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P(S₃)</td>
<td>.1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| EMV for A₁ = 19, for A₂ = 13.5, for A₃ = 13.5 So A₁]
4.9 FURTHER READING


